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Investigating Some Junctures in Relational Thinking: A study of Year 6 and Year 7 Students from Australia and China

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Recently, the study of relational thinking became one of the most attractive problems in mathematics education. One aim of this study in this field is to look at how students make the transition from arithmetic to algebra, and to identify some key linkages in this process. This study uses number sentences involving one and two unknown numbers to identify some key junctures between relational thinking on number sentences and an ability to deal with sentences involving literal symbols. In this paper, the focus is on how students' performances on sentences involving two unknown numbers influenced their performance on sentences involving literal symbols.

Key words: number sentences, relational thinking, categorising of relational thinking.

Rationale for the Study

In their study, "The algebraic nature of students' numerical manipulation in the New Zealand Numeracy Project", Irwin and Britt (2005) argue that the methods of compensating and equivalence that some students use in solving number sentences may provide a foundation for algebraic thinking (p. 169). These authors give as an example the number sentence 47 + 25 which can be transformed into 50 + 22 by adding 3 to 47 and subtracting 3 from 25. They claim (p. 171) "that when students apply this strategy to sensibly solve different numerical problems they disclose an understanding of the relationships of the numbers involved. They show, without recourse to literal symbols, that the strategy is generalisable." Several authors, including Stephens (2006) and Carpenter and Franke (2001), refer to the thinking underpinning this kind of strategy as relational thinking. A deep understanding of equivalence and compensation is at the heart of relational thinking. Students need to know the direction in which compensation has to be carried out in order to maintain equivalence (Irwin & Britt, 2005; Kieran, 1981).

The *direction* of compensation that is appropriate for the operation of addition, for example, is inappropriate for subtraction. Some children reason incorrectly that a number sentence such as 87 – 48 is equivalent to 90 – 45. Failing to recognise that the relationship of subtraction or difference is fundamentally different to addition, these children treat the direction of compensation for subtraction the same as for addition. Other children, however, explain that "in order for the difference to remain the same, the same number has to be *added* to each number in the expression." For these children, 87 – 48 = 89 – 50.

Methodology

Design of Questionnaire

In this study involving students in Year 6 and Year 7, number sentences involving all four arithmetical operations were therefore included. These first took the form of sentences with one missing number where the value of that number can be found either by relational thinking or by computation. These are called Type I number sentences. Examples of such sentences used in the study were as follows:

$43 + \square = 48 + 76$, $39 - 15 = 41 - \square$, $\square \times 5 = 20 \times 15$, $21 \div 56 = \square \div 8$

Students were also asked to write briefly how they found the value of the missing number.

Previous studies (Stephens, 2007; Stephens, Isoda, & Inprashita, 2007) identified a small group of students who appeared to opt for computational methods to deal with missing number sentences, but were quite capable of using ideas of equivalence and compensation to solve sentences involving literal symbols; and were clearly different from those students who could use only computational methods.

The study therefore needed to include a second type of number sentence where students are "pushed" to think relationally. Number sentences involving two unknown numbers, such as 18 + (Box A) = 20 + (Box B), seem to have this potential. These are called Type II number sentences. While it is possible to use computational methods to give particular instances of correct sentences taking this form, identifying a general structural relationship requires students to move beyond computational thinking. For example, a clear relational explanation might say that the above sentence will always be true *as long as the number in Box A is two more than the number in Box B*. Being able to derive a correct mathematical generalisation from numerical examples is a key element of algebraic reasoning (Carpenter & Franke, 2001; Lee, 2001; Zazkis & Liljedahl, 2002).

Type II number sentences involving two unknown numbers – across all four operations – were included in the study. These allowed some scope for computational approaches, but, following Fujii (2003), identifying the critical numbers and the relational elements embodied in these expressions required that students move beyond computation and focus especially on expressing the underlying mathematical structure.

Finally, to determine if equivalence and compensation provide a foundation for algebraic thinking, some questions involving literal symbols were included. Therefore, several questions modelled after the research programme, *Concepts in Secondary Mathematics and Science*, (CSMS, see Hart, 1981) asked students: What can you say about *c* and *d* in the following mathematical sentence?

$$c + 2 = d + 10$$

This third type of question was called a Type III sentence. It allowed students to say that this sentence will be true for *any* values of *c* and *d* provided *c* is 8 more than *d*. But other students may fall short of this, simply giving several values of *c* and *d* for which the sentence is true. Other students may say "*c* is more than *d*" but cannot specify the relationship. The value of questions such as these is that they can be given partial or complete relational interpretations, indicating different stages of development.

A questionnaire, consisting of eight pages, was comprised of these three types of questions with each type distributed across the four operations. Type I comprised missing number sentences involving one unknown number. Type II comprised arithmetical sentence with two unknown numbers. Type III was structurally similar to the second type but explicitly included literal symbols.

Type I: Missing Number Sentences with One Unknown Number

Figure 1 shows Type I questions used for addition on the first page. The position of the box denoting a missing number was varied for each item. The given numbers were chosen so as to provide numbers on either side of the equal sign such that a relational approach to finding the missing number was encouraged. Of course, it was also possible for students to solve each question by computation. After each item, space was provided for students to write how they had found the missing number. Similar Type I sentences were used for the other three operations.

1. For each of the following number sentences, write a number in the box to make a true statement. Explain your working briefly.

23 + 15 = 26 + □	73 + 49 = □ + 47
43 + □ = 48 + 76	□ + 17 = 15 + 24

Figure 1. **Question 1 involving addition and Type I questions.**

Type II and III: Sentences Involving Two Related but Unknown Numbers

Each even-numbered page opened with a Type II question using two boxes, denoted by Box A and Box B, and employing one arithmetical operation. Type II questions are exemplified in parts (a) to (d) in Figure 2. These were then followed by a related Type III question, shown in part (e), involving the same arithmetical operation and literal symbols.

2. Can you think about the following mathematical sentence:

18 + ☐ = 20 + ☐ Box A Box B (a) In each of the sentences below, can you put numbers in Box A and Box B to make each sentence correct? 18 + ☐ = 20 + ☐ Box A Box B 18 + ☐ = 20 + ☐ Box A Box B 18 + ☐ = 20 + ☐ Box A Box B (b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B? (c) If instead of 18 and 20, the first number was 226 and the second number was 231 what would be the relationship between the numbers in Box A and Box B? (d) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly. (e)What can you say about c and d in this mathematical sentence? $c + 2 = d + 10$

Figure 2. **Question 2 involving addition and Type II and Type III questions.**

The three kinds of formats above were used for questions involving the other three operations, with two pages devoted to each of the four arithmetical operations.

Participants

The participants were drawn from Year 6 and Year 7 students in two schools, one in Australia and one in China. The Chinese sample consisted of two intact classes consisting of 32 students in Year 6 and 36 students in Year 7. In the Australian school, one Year 6 class of 25 students was involved and three Year 7 classes consisting of 71 students altogether. The sample was a convenience sample. The performances of students are therefore not presented as being normative of schools in each country, and may reflect the teaching they have received. However, it is possible to examine students' performances on the three types of sentences, and to track what students do over certain junctures. Translation of the questionnaire into Chinese was prepared by faculty members at an Eastern Chinese university. Graduate students at the same university and two Chinese speaking graduates in Australia assisted with the translation of students' responses. Each student's written responses were read independently by two markers. A very high degree of consistency of classification was evident across markers in both countries.

Key questions to be investigated in this Paper

The focus of this paper will be confined to students' responses to Type II and Type III sentences. In a later paper, we will describe students' responses to Type I number sentences and how their computational or relational responses to these sentences influenced their responses to Type II and Type III sentences.

Key questions to be investigated in this paper are the different kinds of responses to Type II and Type III sentences and how performance on Type II sentences influenced performance on Type III sentences.

All students attempted the addition and subtraction questions involving Type II and III sentences. Some Year 6 students in the Chinese school had difficulty going any further, but this provided sufficient evidence. Year 6 students in the Australian school and Year 7 students in both schools generally completed all, or most of, the questionnaire.

Results on Type II and III Sentences

Type II and Type III sentences had been deliberately crafted to "push" students into relational responses even if it was possible for them to complete parts (a) of these questions by computation. Almost all students without exception were able to place numbers correctly in Box A and Box B to make a correct sentence. Some students admittedly chose quite small numbers to place in the boxes to give correct sentences.

Having constructed several correct sentences in this way, all students attempted to describe the relationship between the numbers in Box A and Box B. However, there were clear differences in the way students describe the relationship between the number in Box A and Box B and between c and d. These differences are shown in the following Table 1. Non-Directed relational responses to part b and part c of Type II number sentences, as shown in Table 1, are incomplete but they are not wrong. The same can be said about Directed (no magnitude) Relational responses and Directed (non-referenced) Relational responses. Referenced and Directed Relational responses to part c and d of Type II sentences were so evident in the following: "b is 2 less than a" (Chinese Year 6 student) and "Box A is 3 less than Box B (subtraction)" (Australian Year 7 student).

Table 1

Sstudents' Description of the Relationship between the Numbers in Box A and Box B and between c and d

Response type	Examples
Incorrect Relation	Students continue to use 'difference' on multiplication and division question -the difference (between *c* and *d*) is always 16 (as in $c \div 8 = d \div 24$)

Non-directed Relation	-they would always be **5 apart** [as in 3 ÷(Box A) = 15 ÷(Box B)] -there is always a **3 difference** [as in 72 – (Box A) =75 – (Box B)] -the numbers have **a distance of 2** [as in 18+(Box A) = 20 + (Box B)]
Directed (no magnitude) or Directed(non-referenced*) Relation	-so long as the number in Box B is larger [as in division or subtraction examples above] -*d* will be more than *c* [as in $c - 7 = d - 10$] -one number is always higher than the other number by 2 [as in addition example above] -one is 2 more than the other [as in addition example above]
Referenced Directed Relation	-one is 3 more than the other, Box B is bigger [as in subtraction example above] -*c* is 8 ahead of *d* [as $c + 2 = d + 10$] -A is 5 times less than B [as in 3 ÷(Box A) = 15 ÷(Box B)] -difference of 2, A larger [as in 18 + (Box A) = 20 + (Box B)]

Note. "non-referenced" means not to point out the relational object, such as Box A and Box B, or *c* and *d*.

Students who used incomplete expressions to describe the relationship between Box A and Box B were not able to give a successful response to part d of Type II number sentences which asked "if you put any number in Box A, can you still make a correct sentence?" Example of relational responses to part d questions are "as long as B is 2 less than A" (Chinese Year 6 student answering to the involving condition) and the responses by an Australian Year 7 student to question for part d involving subtraction who said "any number can be in Box A, so long as Box A is 3 less than Box B".

Generic Categorisation for Types of Relational Thinking on Type II and III Questions

Based on the responses to Type II and III sentences, we categorised relational thinking as Established Relational Thinking, Consolidating Relational Thinking, and Emerging Relational Thinking. In both schools in Year 6, many students still appeared to be operating as Emerging Relational Thinkers, less so in the Australian school as shown in Table 2 below. But in both schools by Year 7 the majority of students were able to show Consolidating or Established Relational Thinking. The defining characteristics of each of these three categories are as follows: Consolidating Relational thinkers – as illustrated in Figure 4–are: (a) almost always able to specify the relationship between the numbers in Box A and the numbers in Box B with clear references to the numbers, including the magnitude and direction of the difference between them; (b) sometimes able to give a complete explanation as to how any number might be used in Box A and still have a true sentence; (c) usually able to refer to some feature of the relationship between *c* and *d*, or give a specific pair of values for *c* and *d*, but cannot give a complete explanation of the relationship.

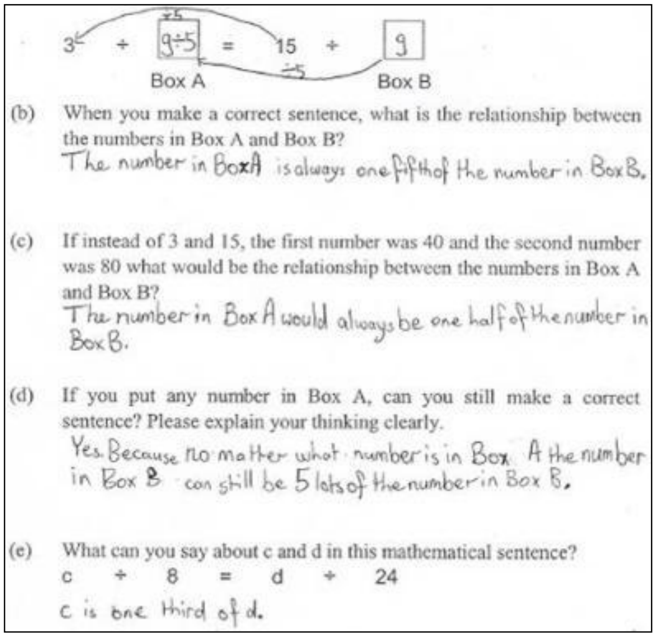

3 ÷ [9÷5] = 15 ÷ [9]
Box A Box B
(b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?
The number in BoxA is always one fifth of the number in BoxB.
(c) If instead of 3 and 15, the first number was 40 and the second number was 80 what would be the relationship between the numbers in Box A and Box B?
The number in Box A would always be one half of the number in Box B.
(d) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly.
Yes. Because no matter what number is in Box A the number in Box B can still be 5 lots of the number in Box B.
(e) What can you say about c and d in this mathematical sentence?
c ÷ 8 = d ÷ 24
c is one third of d.

Figure 3. **Sample response showing established relational thinking.**

Established Relational thinkers – as illustrated in Figure 3 – are almost always able to: (a) specify the relationship between the numbers in Box A and the numbers in Box B with clear references to the numbers, including the magnitude and direction of the difference between them; (b) employ a similar form of words used to describe this relationship as a part of the condition that describes how any number can be used in Box A and still make a true sentence; (c) explain clearly how *c* and *d* are related for the Type III sentence to be true, treating *c* and *d* as general numbers.

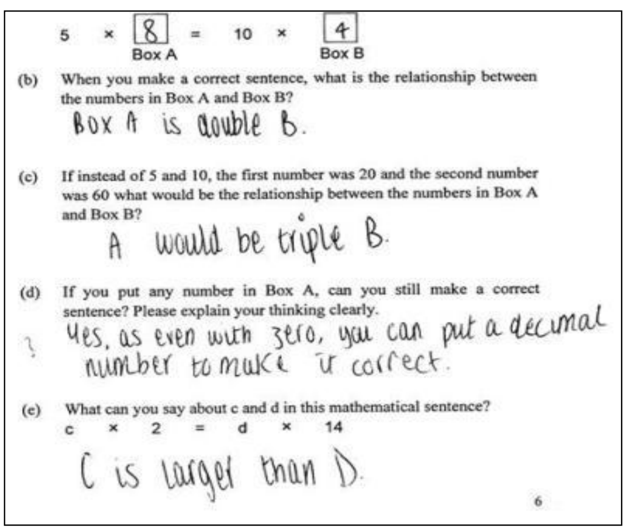
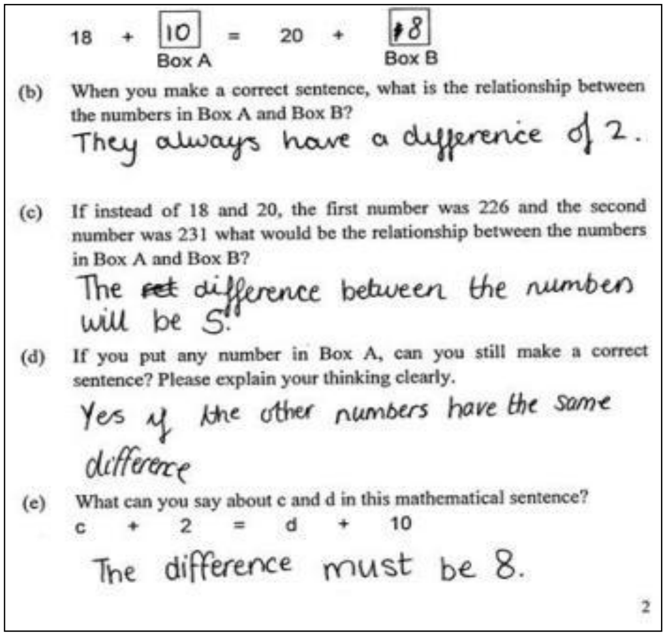

5 × [8] = 10 × [4]
Box A Box B
(b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?
Box A is double B.
(c) If instead of 5 and 10, the first number was 20 and the second number was 60 what would be the relationship between the numbers in Box A and Box B?
A would be triple B.
(d) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly.
Yes, as even with zero, you can put a decimal number to make it correct.
(e) What can you say about c and d in this mathematical sentence?
c × 2 = d × 14
C is larger than D.
6

Figure 4. **Sample response showing consolidating relational.**


18 + [10] = 20 + [8]
Box A Box B

(b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?
They always have a difference of 2.

(c) If instead of 18 and 20, the first number was 226 and the second number was 231 what would be the relationship between the numbers in Box A and Box B?
The ~~set~~ difference between the numbers will be 5.

(d) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly.
Yes if the other numbers have the same difference

(e) What can you say about c and d in this mathematical sentence?
c + 2 = d + 10
The difference must be 8.

2

Figure 5. **Sample response showing emerging relational thinking.**

Emerging Relational thinkers– as illustrated in Figure 5 – typically: (a) identify some feature of the numbers used in Box A and Box B, but do not completely specify the relationship between the numbers used in Box A and Box B; (b) focus on this feature when trying to explain how any number can be used in Box A and still have a true sentence, but again are unable to describe this relationship completely; (c) attempt to give a correct pair of values for which c and d might make a true sentence, or do not attempt this question at all.

Among Established and Consolidating Relational thinkers, shown in Table 2, there was a striking association between making a clear and correct response thinking to part d and describing the relationship between the values of c and d to make the corresponding Type III sentence true. Among the 26 Year 6 students in both schools, classified as Consolidating or Established Relational thinkers, 70% of those who gave a correct response to a part (d) item also correctly described the relationship between c and d in the corresponding part (e) item. For the corresponding group of 89 Year 7 students, in both schools, a successful response to a part d item was followed in 90% of cases by a successful description of the relationship in part e between c and d. On the other hand, in no case was a successful response to a part e item preceded by an inadequate response to its related part d.

Questions involving two unknowns clearly met their purpose of pushing students beyond computation. In addition, there were three Year 6 students and 28 Year 7 students in the Australian school and six Year 7 students in the Chinese school whose responses were

Established Relational for all Type II and Type III questions. Other students gave responses that were consistently Established Relational for those questions they attempted.

Table 2

Comparison of Performances between Chinese and Australian Students

Country	Student		Type II and Type III
Chinese	Year 6	19	13
	Year 7	5	31 [a]
Australian	Year 6	12	3 [b]
	Year 7	13	58 [c]
		Emerging Relational	Consolidating or Established Relational

Note: (a) 6 out these 31 students successfully completed all questions; (b) 3 out of these 13 students successfully completed all questions; (b) 28 out of these 58 students successfully completed all questions.

Analysing the performances of the four different groups does allow several important conclusions to be drawn concerning some key junctures in relational thinking.

Conclusions

In its report *Algebra: Gateway to a Technological Future*, The Mathematical Association of America (2007) recognises that “we still need a much fuller picture of the essential early algebra ideas, how these ideas are connected to the existing curriculum, how they develop in children’s thinking, how to scaffold this development, and what are the critical *junctures* of this development” (p.2). This study is about exploring some of those junctures, connections or linkages in relation to Type II and Type III sentences.

Different student responses to part b and c of Type II questions have different potential for completing part d relating to Type II and part e relating to Type III questions. If students did not completely specify the numbers in Box A and Box B and the relationship between them in parts b and c, they were *unable* to specify a condition in part d that describes how any number might be used in Box A and still have a true sentence. It may be theoretically possible for a student to give an incomplete answer to part b (and part c) of Type II questions, and still give a correct and complete response to part d. But, from a linguistic (language and thinking) perspective, if a student answers part b (and part c) incompletely, it appears to be very difficult to rectify this incompleteness in part d when asked to specify a condition that describes how any number might be used in Box A and still have a true sentence.

The expression ‘any number’ is intended to denote a general number. Established Relational Thinkers were able to use this interpretation. It is possible that some Consolidating and Emergent Relational Thinkers may give a different interpretation to the expression ‘any number’ as denoting an ‘extreme’ or ‘peculiar’ number outside their expected range of variation (e.g. true only for positive whole numbers). These students may therefore be unsure if ‘any number’ can be used in Box A and still have a correct sentence. This question could be pursued further in interviews.

A successful generalisation about Type II number sentences is almost always followed by a successful explanation of the relationship between *c* and *d* in Type III sentences. It is true that some students who successfully completed part e were not able to complete part d successfully. These students may see part e as a more familiar textbook question, and they appear unable to see a structural similarity between part d and part e. Almost all students who successfully completed part d also completed part e successfully, suggesting that they were able to see some structural similarity between the two questions

A more striking conclusion is that students who did not completely specify the numbers in Box A and Box B and the relationship between them in parts b and c for Type II sentences were *never* able to specify a condition that describes the relationship between *c* and *d* in Type III sentences (question e). This suggests that Emergent Relational Thinking which is either Non-directed, or Directed with no magnitude, or Directed with no reference, as evidenced in responses to part b and c, also prevents students from dealing successfully with the kind of generalisation required for parts d and e.

Established Relational Thinkers seemed able to refer to *c* and *d* as general numbers when they used words to describe a condition for the sentence involving *c* and *d* to be true, such as ‘As long as *c* is 8 more than *d* ’. When students wrote symbolic expressions for this condition such as $c - d = 8$ or $c = d + 8$, it seemed highly likely that they were referring to c and d as general numbers. (This assumption might be further investigated using interviews).

Even when logical qualifiers such as ‘always’, ‘needs to’ and ‘must’ are used in non-directed Relation, Directed (no magnitude) Relation, and Directed (non-referenced) Relation, their use does not help students to generalise what would happen if any number was used in Box A, or to correctly describe the relationship between *c* and *d*. When these logical qualifiers are used by Established Relational Thinkers, they tend to form a consistent mathematical discourse in responses to Type II and Type III questions.

Implications for Teaching and Learning

Mathematical discourse used by Emergent Relational Thinkers seems to restrict their ability to think about variables. It may be that when these students come to articulate what they are aware of, their attention is diverted to a part or single feature of the relationship rather than some more comprehensive whole. These students need to be helped to identify the quantities that vary and to describe the relationship. Many may not be familiar with the kinds of relationships and the language used to describe the relationships which prove to be *productive* in mathematics. A key issue is to help all students develop a mathematical language that supports a functional thinking about variables. Non-directed relations, Directed (no magnitude) relations and Directed (non-referenced) relations may appear adequate in response to questions b and c; however these ‘limited’ relational descriptions seem to ‘lock’ students into a form of discourse that does not promote the ability to successfully generalise.

While these descriptions denote an early stage of relational thinking development, teachers should be aware that they need to help their students articulate referenced and directed relational descriptions. This may be done through highlighting to students the disadvantages and advantages that different descriptions offer.

Many Consolidating Relational Thinkers believe that the relationships hold only for a specific range of numbers, e.g. positive whole numbers. Likewise, many of these students give one specific pair of values for the relationship between *c* and *d*. Teachers need to show that the permissible range can include rational numbers and negative numbers for both Type II and III sentences. These two kinds of sentences, when used in parallel, have a clear potential for developing and refining algebraic thinking. This may be exploited by asking students first to describe carefully the specific features of the Type II number sentences and then to apply these features to the structurally similar corresponding Type III sentences. That is what Established Relational Thinkers are able to do on their own, but with explicit teaching many more students can be helped to think that way too.

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