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"Zahlenblickschulung" as Approach to Develop Flexibility in Mental Calculation in all Students

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Flexibility in mental calculation is a highly pursued goal in elementary math education, and the presented qualitative study focuses on its development in 20 first grade students from different instructional contexts who have difficulties in learning arithmetic. The crucial question was how the approach "Zahlenblickschulung" supports the development of flexibility in mental calculation especially in this group of students. The paper is structured in two parts: In the first part, the term "Zahlenblick" is defined based on the consideration of the different notions of number sense and structure sense. Furthermore, the approach "Zahlenblickschulung" is explicitly described and illustrated by specific examples. The second part of the paper is dedicated to present the methods of data collection and data analyses, as well as the essential results of the study. In this study, the students from different German classrooms have been interviewed three times during first grade and one time at the beginning of second grade. The interviews focused on students' solution strategies and argumentation strategies, and data was qualitatively analyzed with the same focus. The results suggest "Zahlenblickschulung" as an instructional approach that supports recognition of number patterns and numerical relationships. Furthermore, they indicate that this recognition is a crucial requirement not only for developing flexible and adaptive expertise, but also for learning to calculate itself (beyond counting).

Keywords: flexibility, mental calculation, less advanced students.

In the last two decades, flexible mental calculation has been increasingly considered as an important goal in elementary math education (Lorenz, 1997; Selter, 2000; Verschaffel, Luwel, Torbeyns, & Dooren, 2009). Moreover, various research with different aims have been conducted in this field. Distinct studies revealed that "Zahlenblickschulung"[1] supports the development of flexible mental calculation. Those studies predominantly

[1] We keep the German term "Zahlenblick" because the concept that is connected with that term gets lost in translation and cannot be expressed with the linguistically related term "number sense".

focused on students from second and third grade (Grüßing, Schwabe, Heinze, & Lipowsky, 2013; Heinze, Marschik, & Lipowski, 2009; Heinze, Schwabe, Grüßing, & Lipowsky, 2015; Rathgeb-Schnierer, 2006). There is hardly any research regarding the development of flexible and adaptive expertise in first grade students. Additionally, there is still no satisfactory answer to the question at what time flexibility in mental calculation should be supported and developed: Is it after students have developed basic abilities in mental calculation or is it with the beginning of first grade (Threlfall, 2009; Verschaffel, Torbeyns, De Smedt, Luwel, & van Dooren, 2007)? In addition, there is still research needed to answer the question whether less advanced students are able to develop flexible mental calculation or not (Geary, 2003; Peltenburg, van den Heuvel-Panhuizen, & Robitzsch, 2011). With its aim and methodological approach, the presented project tries to bridge the described gap.

Theoretical Framework

The recognition of problem characteristics and numerical relationships is a crucial prerequisite for flexible and adaptive expertise (Rathgeb-Schnierer, 2006; Threlfall, 2009). “Zahlenblickschulung” is a special approach for mathematics education that puts the emphasis on number patterns and numerical relationships. Within the theoretical framework, this approach is described in detail as well as the notion of flexibility that underlies this study.

“Zahlenblickschulung”

The approach “Zahlenblickschulung” pursues the aim to develop “Zahlenblick”. To clarify what “Zahlenblickschulung” means, we describe the meaning of “Zahlenblick” and its related concepts like number sense and structure sense.

Number sense, structure sense and “Zahlenblick”. The concept *number sense* is defined in two distinguished ways; as result either of experienced based development or as inherent skill.

> “With respect to its origins, some consider number sense to be part of our genetic endowment, whereas others regard it as an acquired skill set that develops with experience.” (Berch, 2005, p. 333)

Regarding the construct *structure sense,* the notions are quite similar. Lüken’s definition (2010) of early structure sense is reminiscent of inherent competence, whereas Linchevski and Livneh (1999) point out the necessity of its development.

“Zahlenblick” is a result of development, and means the competence to recognize problem characteristics, number patterns and numerical relationships immediately, and to use them for solving a problem (Schütte, 2004a). Comparing number sense, structure sense and “Zahlenblick”, it is obvious that the meaning of number and structure sense as acquired skill is quite similar to our notion of "Zahlenblick", which can be developed by specific activities. Since there are still discussions about the different definitions, we keep the term "Zahlenblick" in the previously described sense by Schütte (2004a) (see above).

Development of "Zahlenblick". For supporting the development of "Zahlenblick", it is crucial to provide activities that highlight problem characteristics, patterns and numerical relationships (Rechtsteiner-Merz, 2013; Schütte, 2004a). For elementary students, these activities target the development of number concepts, understanding of operations and strategic means[2]. A crucial feature of activities that promote the development of "Zahlenblick" is their emphasis on number patterns and numerical relations. Therefore, students are purposely encouraged to *recognize* number patterns, problem characteristics and relations between numbers and problems, and to *sor*t and *arrange* problems by using structural relations. In addition to this specific emphasis in terms of content, all activities include cognitively challenging questions to provoke students' thinking and reflection. By combining mathematical topics with challenging questions, the teacher encourages the increase of metacognitive competences (Rechtsteiner-Merz, 2013). Overall, activities that promote the development of "Zahlenblick" differ from common activities in number and operation since they do not emphasis to solve a problem in the first place, but rather focus on problem characteristics, patterns and numerical relationships. Therefore, many activities include addition and subtraction problems with the aim to give students the chance to discover and discuss inherent structures and relations. To clarify the meaning of "Zahlenblickschulung", we give three different examples for typical activities:

(1) One visual number pattern – various options for interpretation: This activity focuses on various numerical interpretations of one visual number pattern based on different perspectives (Figure 1). For instance, looking at the grey and transparent dots separately a suitable interpretation of the visual number pattern in Figure 1 would be 13=10+3, but it could also be 13=5+5+3. Whereas, focusing on the rows a possible interpretation would either be 13=6+7, 13=6+6+1 or be 13=5+1+5+2. Each single interpretation is closely linked to a specific perspective. Within this activity, the central aim is to encourage students to discover various addition problems and discuss their point of view with each other and the teacher (Schütte, 2004b). From this activity, students develop strategic means by gaining a deeper understanding of the part-

[2] Strategic means are distinct devices to modify problems to make them easier. They can be flexibly combined in a solution process, and include for instance composing and decomposing, modifying a problem, deriving the solution from a known fact, and using analogies (e.g. Rathgeb-Schnierer, 2006; Rathgeb-Schnierer, & Green, 2013)

whole-concept, getting insight in various partitions, and learning to reason about their thinking.

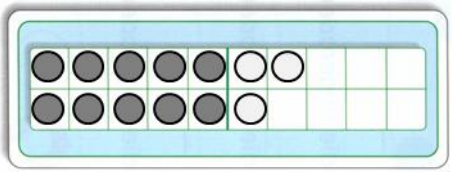

Figure 1. *Visual number pattern.*

(2) Sorting the problems: This activity emphasizes on sorting addition and subtraction problems up to 20 in categories like “I need to count it”, “I know it” and “I know a trick to solve it” (Rechtsteiner-Merz, 2011). When introducing the activity, students were encouraged to sort each single problem into a suitable category without solving it. Therefore, they need to carefully look at each problem and focus on problem characteristics as well as numerical relations. All three categories are subjective, and lead to different sorting results among the students. Nevertheless, the students can talk about their sorting, and compare their problems in each category. While discussing with other students and the teacher and comparing their sorting, they might discover new patterns, new relations or new strategic means. This activity aims to reflect about problem characteristics on a more general – algebraic – level, for instance: What is the reason for assigning several different problems to one category? Do all problems in one category have similarities? Are there some problems in the category “I need to count it” which are similar to those in the category “I already know a trick”? Is it possible to use problems already known by heart to facilitate the solution of problems from the other categories?

(3) “Problem-family”: A “problem-family” is described as a set of structurally related problems (see Figure 2). This activity intends to put the emphasis on structural relations between a set of problems. (Rechtsteiner-Merz, 2011). The students start with one problem (e.g. 5+5=10) and arrange many cards with related problems (e.g. 5 + 6, 6 + 6, 4 + 6 etc.) around this first one with the aim to make the relations visible (Figure 2). Subsequently, the students were encouraged to describe their arrangements, and give reasons for their decisions. This activity does not focus on solving problems, but on recognizing problem features and relationships. Within this activity, students discover relations between problems; this is a relevant prerequisite for going beyond counting and using derived fact strategies.

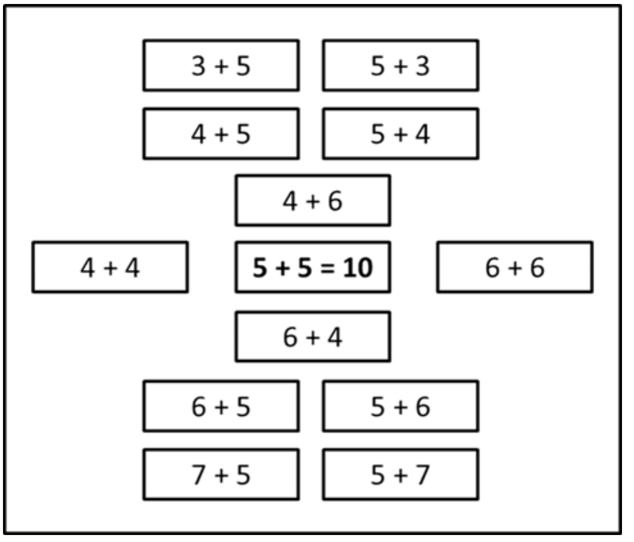

Figure 2. *A "Problem-family" (Rechtsteiner-Merz, 2013, p. 113).*

The depicted activities were part of the project described below, and therefore, specifically described for the use in first grade. However, all activities can be modified and applied for all elementary grades. They can also be assigned to other number systems than natural numbers, and therefore be introduced in secondary school as well.

Flexible Mental Calculation

According to Rathgeb-Schnierer and Green (2013), we define mental calculation as "solving (multi-digit) arithmetic problems mentally without using paper and pencil procedures" (Rathgeb-Schnierer & Green 2013, p. 553; brackets are added by the authors).

Regarding flexibility in mental calculation, there are different definitions in current literature (e.g. Rathgeb-Schnierer & Green, 2013; Threlfall, 2009; Verschaffel et al., 2009). The consensus among all these definitions is the idea that flexibility in mental calculation includes two aspects: the knowledge of different solution methods and the ability to adapt them appropriately when solving a problem. The differences in the definitions refer exactly to the second aspect: the adaptive use of solution methods (Nunes, Vargas, Dorneles, Lin, & Rathgeb-Schnierer, 2016; Rechtsteiner-Merz, 2013, 2015; Rechtsteiner-Merz & Rathgeb-Schnierer, 2015; Verschaffel et al., 2009). Systematical analyses of literature have revealed different approaches regarding the meaning of "adaptive use of solution methods", and how this can be operationalized (Rechtsteiner-Merz, 2013). Generally, the different notions can be classified in two main perspectives:

- First, the adaptive use of solution methods is understood as match of solution methods and problem characteristics. Those approaches are based on the assumption that the one most suitable solution method exists for each specific problem (task), and that this method is chosen consciously or unconsciously (see e.g., Blöte et al., 2000, 2001; Klein, & Beishuizen, 1998; Star & Newton, 2009). Two different ways to judge if a solution method matches the problem characteristics can be found in these research approaches: (1) accuracy and speed of obtaining a solution (Torbeyns, Verschaffel, & Ghesquière, 2005; Verschaffel et al., 2009), and (2) number of solution steps (Star & Newton, 2009).

- Second, the adaptive use of solution methods is judged in connection with the cognitive elements that underlie the solution process (Rathgeb-Schnierer & Green, 2013; Threlfall, 2002; 2009). Researchers who adopt this approach (see e.g., Rathgeb-Schnierer, 2010; Rathgeb-Schnierer et al., 2015; Threlfall, 2002; 2009) focus on the match between the combination of strategic means and the recognition of number patterns and numerical relationships of a given problem during the computation process. This recognition depends on students' knowledge of numbers and operations. To identify cognitive elements that underlie a solution process, it is necessary to focus on the processes itself and reveal if students rely on recognized characteristics and numerical relations or learned procedures.

In this project, we define flexible mental calculation as flexible and adaptive use of solution methods when solving arithmetic problems. Hereby, flexibility is understood as the ability to switch between solution methods, and acting adaptively means that the solution process matches recognized problem characteristics, number patterns and numerical relationships.

Methods

Questions and Design

The project focuses on learning processes of less advanced students, and investigates whether less advanced students are able to develop flexible mental calculation when educated with "Zahlenblickschulung" in first grade. Based on the theoretical background, our study aimed to answer the following research question: Are first graders with difficulties in learning numbers and operations able to develop flexible mental calculation when educated with "Zahlenblickschulung"?

To examine the benefit less advanced students draw from the approach "Zahlenblickschulung" a qualitative study was designed that focuses on learning processes. The study included two parts: the instructional approach and the investigation of learning processes (Figure 3).

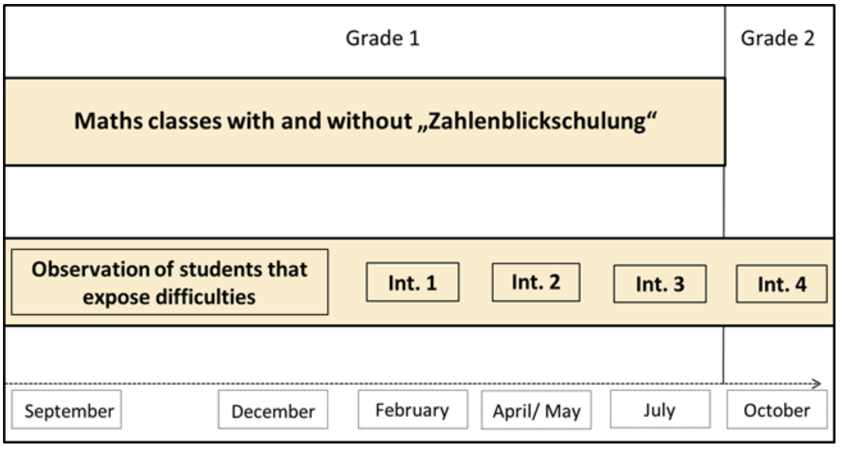

Figure 3. *Design* (Rechtsteiner-Merz, 2013, p. 153).

Participants and Classroom Practice

Interviews were conducted by one of the researchers. Interview groups consisted of 20 less advanced students from the eight different classes in first grade, 11 boys and 9 girls (6-8 years old). These students were purposely chosen based on classroom observations over 6 to 8 weeks combined with two different non-standardized tests. Based on Schipper’s (2005) claim that approximately one out of five students from each classroom develops problems in learning mathematics, we decided on the size of the sample. From the whole sample, twelve students (five classes) attended math classes with the approach “Zahlenblickschulung” (in one of four math lessons per week at the minimum) (see above); eight students (three classes) attended best practice math classes, but without the approach “Zahlenblickschulung” (Figure 3).

All teachers assigned to the approach “Zahlenblickschulung” were experienced in this field. Additionally, teachers were counseled by one of the researchers four times during the academic year, and received a very detailed manual with descriptions of all activities and with suggestions for a weekly schedule. Each teacher wrote a logbook documenting all activities conducted during a week. The teachers without the special approach “Zahlenblickschulung” were participants of a specific nation-wide in-service teacher-training program in mathematics education (so called SINUS program; Prenzel, Friedrich, & Stadler, 2009). By choosing teachers from these program we could assure to include only best practice math classes in our study.

Data Collection

From January of first grade to October of second grade, four semi-structured interviews were conducted with all students (80 interviews in total) by the same interviewer (Figure 3).

The interviews 1 – 3 took place during first grade and included addition problems with single-digit numbers. The last interview, scheduled at the beginning of second grade, contained addition problems up to 100 (Table 1). In Germany, numbers up to 100 are introduced in second grade, and at the beginning of second grade, students are not familiar with two-digit addition problems.

Table 1

Addition Problems for Interview 1 – 4 (Rechtsteiner-Merz, 2013, p. 186)

Interview 1	Interview 2	Interview 3	Interview 4
3 + 3	3 + 3	3 + 3	3 + 3
	6 + 6	6 + 6	6 + 6
7 + 7	7 + 7	7 + 7	7 + 7
	9 + 9	9 + 9	9 + 9
(8 + 8)	(8 + 8)	8 + 8	8 + 8
5 + 6	5 + 6	5 + 6	5 + 6
8 + 7	8 + 7	8 + 7	8 + 7
	7 + 8	7 + 8	7 + 8
(7 + 6)	(7 + 6)	7 + 6	7 + 6
5 + 3	5 + 3	5 + 3	5 + 3
5 + 7	5 + 7	5 + 7	5 + 7
	7 + 5	7 + 5	7 + 5
(8 + 5)	(8 + 5)	8 + 5	8 + 5
		5 + 8	5 + 8
9 + 6	9 + 6	9 + 6	9 + 6
9 + 3	9 + 3	9 + 3	9 + 3
	(3 + 9)	3 + 9	3 + 9
4 + 9	4 + 9	4 + 9	4 + 9
(9 + 8)	(9 + 8)	9 + 8	9 + 8
			25 + 3
			47 + 7
			4 + 91
			69 + 6
			18 + 7
			35 + 8

In all interviews, students were first asked to sort problems into the categories “easy” and “hard” and to reason about their sorting. In the second step, they were encouraged to solve the problems categorized as “easy”, and some of those that were judged as “hard”. After each solution process, the students were requested to explain their solution method. Since the standard algorithms are part of the curriculum in third grade, second grade students solve arithmetic problems mentally without using paper and pencil procedures.

All four interviews were scheduled with an interval of at least two months. During the time between the interviews, the students experienced math lessons either with or without special activities for “Zahlenblickschulung”. Based on the specific design and the same interview conditions for all participants, we assured that the slightly possible influences on learning processes are the same for the whole sample.

All interviews were transcribed in a text format, and snap-shots from the videos were added to illustrate the sorting process (Rechtsteiner-Merz, 2013).

Data Analysis

The basic notion of flexibility in mental calculation influenced the data analysis. Based on the method of qualitative content analyses (Mayring, 2008) two different coding systems were developed: A deductive coding system was used for analyzing the solution process. For a systematical identification of the cognitive elements that sustain students’ solution processes, an inductive coding system was developed (Rechtsteiner-Merz, 2013). After a separated analysis in the first place, both coding systems were combined in the second step according to Kelle and Kluge (2010). Based on this combination, types could be constructed, and a typology of flexible mental calculation in first grade could be developed (Rechtsteiner-Merz, 2013; Rechtsteiner-Merz, & Rathgeb-Schnierer, 2015).

Results

The data analysis led to a typology about flexible mental calculation (Figure 4) and revealed four central hypotheses (Rechtsteiner-Merz, 2013).

Figure 4 shows the empirically evolved types in a two-dimensional graph: The vertical axis represents the decrease of counting and increase of calculation: the horizontal axis illustrates the increase of reliance on numerical relationship in argumentation.

Four main and five temporary types were derived from data: Main types represent an endpoint of students’ development at beginning of second grade, temporary types characterize phases of transmission. Consecutively, we outline the main types: Students with *counting strategies*[3] solved each problem by counting, usually beginning with the large number. For example, Tim solved the problem 7 + 8 by switching the terms of the sum. Then, he took his fingers, showed eight and counted “9, 10, 11, 12, 13, 14, 15 (*showing each number with the fingers*)”. Students with *consistent use of procedural mastery* were able to

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[3] For calculating, you can use different tools for solution: counting, basic facts, strategic means. Counting can be distinguished if it’s with or without models and in there are counting-all or counting-on strategies used (Carpenter, & Moser, 1982).

solve problems up to twenty predominantly by calculating. Therefore, they always exhibited the same solution procedure without referring to any problem characteristics. They argued for instance: "I do it always like this" or "like always I go up to ten and then the rest". Students who exhibited *partly basic facts with relational expertise* used different strategic means and relied on problem characteristics. They were able to describe the solution process and gave reasons for their strategic means in an elaborate way like in the following example: "These problems are easy (*points to* 8+5 *and* 4+9), because here it's one less and here it's one more (*points to* 4 *and* 9)". Students who depicted *basic facts extended with relational expertise* relied on basic facts with addition problems up to twenty. Additionally, they were able to solve two digit problems with numbers higher than twenty based on recognized characteristics and numerical relationships (even if this is not a topic in first grade). For example, Lars argued in the last interview regarding the problem 91 + 4: "90 is a high number but if this here is small number (*points at 1 and 4*) I can find a solution". This type is unique for first grade since all addition problems up to twenty can be memorized by heart.

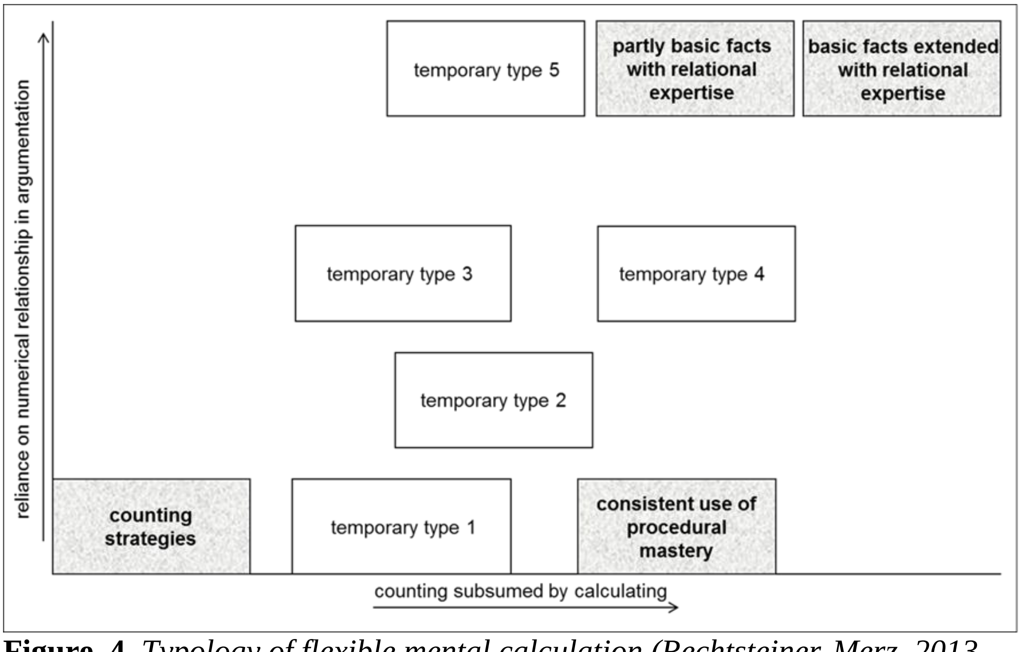

Figure. 4. *Typology of flexible mental calculation (Rechtsteiner-Merz, 2013, p. 243)*

Table 2 shows students' distribution between the different types over all interviews. Each column represents one single type (e.g., CUPM represents the consistent use of procedural mastery, TT 1 stands for temporary type 1), and each line one of the four interview sessions. The numbers in the cells represent the amount of students who were assigned to the specific type. The first number refers to students without "Zahlenblickschulung", the second number to students with “Zahlenblickschulung”. Regarding the assignments to the various types, it is obvious that only students who experienced “Zahlenblickschulung” developed adaptive and flexible expertise (type PBFRE and BFERE). Additionally, the table shows that almost all students who belong to the group with “Zahlenblickschulung” developed computational abilities that go beyond counting, whereas students from the other group predominantly stuck in counting.

Table 2

Students Assignment to Different Types in Different Interview Sessions

	CS	CUPM	PBFRE	BFERE	TT 1	TT2	TT3	TT4	TT5
Int. I	3/3*	0/0	0/0	0/0	4/6	0/0	0/1	0/0	0/3
Int. II	3/2	0/0	0/3	0/0	4/1	0/0	0/3	1/2	0/1
Int. III	2/0	0/1	0/5	0/1	4/2	1/0	1/0	0/0	0/3
Int. IV	2/0	1/1	0/5	0/2	4/0	1/0	0/1	0/0	0/3

Counting strategies = CS, consistent use of procedural mastery = CUPM, partly basic facts with relational expertise = PBFRE, Basic facts extended with relational expertise = BFERE, temporary type = TT; *the numbers which are mentioned second show the children with “Zahlenblickschulung”

These data and the description of students’ individual development enabled us to derive four different hypotheses described below.

Developed Hypotheses and Discussion

(1) Relying on numerical relationships is an absolute condition for developing calculation strategies that go beyond counting.

Different studies have shown that the insistent use of counting strategies is an obstacle for developing calculation strategies that go beyond counting, and students need to develop strategic means to replace counting strategies (e.g. Gaidoschik, Fellmann, & Guggenbichler, 2015; Geary, & Hoard, 2001; Gray, 1991). At the same time, Gaidoschik (2010) identified a group of students who predominantly count, but sometimes refer to strategic means or know facts. Observing their development, it is notable that some of these students overcome counting, and the others remain in counting. Our data clearly suggests that knowledge of basic facts and strategic means is insufficient for the development of a deep understanding of calculation that goes beyond counting; therefore, the focus on numerical relationships and structures is essential. All students who overcame counting strategies (those with as well as those without “Zahlenblickschulung”) relied on numerical relationships at least in one stage of their individual development (*temporary type 3* or *temporary type 4*). On the other hand, all students who showed *predominantly counting relying on procedures* (temporary type 1), remained in this stage and could not progress; this stage seems to be a ‘dead-end road’. Thus, the recognition and use of number patterns and numerical relations appears as a crucial prerequisite for going beyond counting.

(2) "Zahlenblickschulung" supports the development of flexibility in mental calculation.

Students who experienced math education without "Zahlenblickschulung" showed a clear pattern in their development. All of them remained in counting or exhibited only procedural mastery. In contrary, students that experienced the approach "Zahlenblickschulung" showed a gradually increase in flexibility during the period of the study. They mostly exhibited relational expertise at the beginning of second grade. This result underpins the assumption that the approach "Zahlenblickschulung" supports the development of flexibility in mental calculation.

(3) "Zahlenblickschulung" is a fundamental condition for developing calculation strategies and flexibility in mental calculation.

Different studies (e.g. Torbeyns et al., 2005) indicate that middle and high-achieving students develop several number patterns and numerical relationships for going beyond counting independently, whereas under-achieving students do not. Under-achieving students urgently need support for going beyond counting, and to develop flexible mental calculation by a specific approach to math education. Our data analyses suggest that under-achieving students benefit eminently from the approach "Zahlenblickschulung" in two different ways: First, to overcome counting, and second, to develop an appropriate degree of flexibility in mental calculation.

(4) "Zahlenblickschulung" supports the development of conceptual knowledge.

Conceptual knowledge defined as knowledge about generalized facts and principles (e.g. Baroody, Feil, & Johnson, 2007; Star, 2005). Based on this notion, we considered two different aspects as indicators for conceptual knowledge:

- First, students' ability to reason about their solution methods,
- Second, students' capability to transfer the number concepts and solution methods from a well-known range of numbers (up to 20) to a range of numbers (up to 100) not yet introduced at school.

Our data analysis evidenced a big difference between students taught based on the approach "Zahenblickschulung" and those who were not. Students who were not in the sample who experienced "Zahlenblickschulung" were predominantly not able to argue consistently by using mathematical concepts and numerical relations. In contrast, students who were included in the approach "Zahlenblickschulung" usually based their reasoning on conceptual knowledge, and were sometimes able to transfer the knowledge and argumentative structure to problems with higher numbers.

In summary, our results give evidence for the impact of the approach "Zahlenblickschulung" on less advanced students' development of arithmetical expertise. The approach "Zahlenblickschulung" can support all students in developing solution methods that go beyond counting, and gaining flexibility in mental calculation.

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