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Fostering Children's Adaptive Use of Mental Arithmetic Strategies: A Comparison of Two Instructional Approaches

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The adaptive use of strategies to solve problems efficiently is an important goal of arithmetic education. In this article, we examine conceptual and empirical literature related to arithmetic content and pedagogy about students' adaptive use of strategies. Theoretically, two distinct instructional approaches to foster arithmetic strategies have been identified: an explicit approach emphasizing teacher-directed instruction and an implicit approach based on child-initiated inventions. We compare both instructional approaches by contrasting their assumptions, their primary focus, and their expected effects on student learning. Research results from each approach are examined, and we suggest several avenues of useful research on mental arithmetic pedagogy.

Keywords: adaptive use of strategies, strategy flexibility, instructional approaches, multi-digit addition, subtraction

The adaptive use of strategies in arithmetic, that is, solving computation problems efficiently by flexibly applying "advantageous" strategies, is considered an important aspect of mathematics education. To date, the digit-based[1] standard (written) algorithms for the basic arithmetic operations still play a prominent role in many countries, but elementary school curricula also emphasize other number-based strategies (e.g., jump strategy,

[1] The standard algorithms are usually classified as digit-based strategies because they operate on the digits of the two numbers and ignore the place value of these digits during the calculation process. Other strategies, like for example the jump strategy, are classified as number-based strategies because they operate on the numbers and respect the place value.

split strategy, compensation strategy, indirect addition) before the algorithms are introduced (Mullis et al., 2016; Verschaffel et al., 2007). Building on the individual and intuitive ways of approaching arithmetic problems, students should develop the ability to solve a given arithmetic problem with an efficient strategy instead of simply applying the same strategy for all problems. In contemporary views of mathematics education, fostering students' adaptive use of strategies contributes to the important goal of developing conceptual understanding in arithmetic (e.g., Baroody, 2003; Verschaffel et al., 2007). In addition, the adaptive use of internalized number-based strategies allows students to solve specific types of multi-digit arithmetic problems by purely mental calculation without paper-and-pencil computations.

Although the adaptive use of strategies has already been implemented in textbooks and teacher education in several countries for about 20 years, empirical studies have repeatedly revealed elementary school students' low flexibility in applying different strategies (e.g., Csíkos, 2016; Heinze et al., 2009; Hickendorff, 2020; Torbeyns et al., 2009; Torbeyns & Verschaffel, 2016). Findings from Hickendorff (2018) indicate that even if students get explicit instructions in a test to look for a "clever strategy," the adaptive use of strategies only increases slightly. This increase is mostly restricted to addition problems and high achievers and does not result in more accurate solutions than the use of less efficient strategies. Though these findings indicate that the acquisition of skills for adaptive use of strategies is challenging, cross-sectional studies suggest that students' adaptive use of strategies is influenced by the mathematics classroom (Hickendorff, 2020; Sievert et al., 2019; Torbeyns et al., 2017). Accordingly, the understanding of how students acquire these skills and how effective instruction can be organized are important goals of research in this context.

In this article, we review published work on the effectiveness of instructional approaches to teach the adaptive use of strategies in the early elementary school years. Since there are manifold ways of teaching arithmetic strategies and their adaptive use, we examine work that contrasts two ideal-typical instructional approaches representing, on the one hand, a teacher-directed approach that *explicitly* introduces strategies and their adaptive use to students (explicit approach), and on the other hand, a student-centered approach which addresses the adaptive use of strategies *implicitly* by providing opportunities to students to invent strategies (implicit approach). Based on a review of conceptual literature, we present the central ideas of these ideal-typical approaches, aspects of their theoretical background, and possible limitations. Subsequently, we present an overview of the (limited) empirical evidence for these theoretical assumptions by reviewing the existing (quasi-)experimental studies. Finally, we discuss the state of the art and provide ideas for future research. Since we are interested in how students learn the adaptive use of strategies, we restrict our presentation to the early elementary school years and the basic operations of addition and subtraction.

Fostering Students' Adaptive Use of Strategies

From a theoretical perspective, two fundamentally different and ideal-typical instructional approaches for teaching arithmetic strategies and their adaptive use can be distinguished. In the first approach, which we call the *explicit approach*, the teacher introduces the relevant strategies to the students and gives opportunities to practice so that the children can execute the strategies quickly and accurately. In addition, the teacher presents information about task-specific criteria, that is, information about problem types for which specific strategies provide an efficient solution. In the second approach, the *implicit approach*, students are asked to invent arithmetic strategies on their own and subsequently get the opportunity to accumulate individual experience about strategy efficiency by solving given arithmetic problems. In this approach, it is assumed that students develop a sense for adaptive use of strategies based on these learning activities without specific input from the teacher. Hence, the teacher does not present additional (undiscovered) strategies or criteria for efficient strategies for a given problem to the students. Moreover, the teacher avoids setting socio-mathematical norms by characterizing particular strategies as important or communicating her or his preference for certain strategies.

The teaching approaches can be considered as ideal-typical approaches with pedagogically distinct origins (teacher-directed instruction vs. student-initiated inventions). They are based on different assumptions about students' skill acquisition regarding the adaptive use of strategies. Both approaches can be implemented in various ways by, for example, using different representations (e.g., the empty number line versus a purely symbolic representation) or providing different opportunities to compare strategies (e.g., in teacher-centered whole class sessions versus small group sessions with so-called "math conferences", see Selter, 1998).

The explicit as well as the implicit teaching approach are based on specific theoretical frameworks which we will briefly outline below. Before that, we give a short overview of relevant aspects of students' knowledge and skills for the adaptive use of strategies.

Necessary Knowledge and Skills for Adaptive Use of Strategies

Knowledge and skills relevant to students' adaptive use of strategies are characterized by different models in the research literature (e.g., Baroody, 2003). For example, the model on strategy competence by Lemaire and Siegler (1995) described four dimensions: (a) strategy repertoire (knowledge of strategies), (b) strategy distribution (knowledge of how frequently strategies are used for certain problem types), (c) strategy efficiency (skills in accurate and fast execution of strategies), and (d) strategy flexibility (skills in flexibly using different strategies). This model was developed in the context of the strategy-choice assumption, that is, students solve arithmetic problems by selecting a strategy from their strategy repertoire.

The strategy-choice assumption was criticized as too restricted because it ignores the possibility that students generate a strategy during the solution process for a problem (e.g., Baroody, 2003; Threlfall, 2009). Consequently, alternative models on students' adaptive use of strategies comprise not only strategies but also specific arithmetic knowledge or skills. For example, the model presented in Rathgeb-Schnierer and Green (2019) described (a) *tools for solution* which refer to knowledge and skills concerning counting, basic facts, and strategic means as well as (b) *cognitive elements* comprising the application of procedures and the recognition of problem characteristics, number patterns and relationships. Here, strategic means are not ready-made strategies but elements like composing/decomposing (e.g., 26 + 24 = 20 + 20 + 6 + 4), decade analogies (e.g., use 3 + 4 = 7 to solve 23 + 4 =) or transforming (e.g., 6 + 8 = 7 + 7 =) which can be flexibly combined to reduce the complexity of a given problem (Rathgeb-Schnierer & Green, 2019).

The Explicit Approach: Coherence and Adaptive Use

The characteristic feature of the explicit approach is the teacher-directed introduction of relevant strategies and criteria for strategy use. These teaching activities are combined with practice phases during which students work on exercises. These practice phases can be organized in different ways, for example, when students work on exercises to train their fluency in strategy execution, or they work on different types of problems to compare the efficiency of strategies. The theoretical background for this approach is twofold and integrates ideas from scaffolding as a teaching method with those of instructional coherence as a method of structuring the learning content.

Scaffolding is considered an important element of supporting students' learning, especially for complex content in students' zone of proximal development (Vygotsky, 1978). According to van de Pol et al. (2010), scaffolding is an interactive process between teacher and students and is characterized by contingency, fading, and transfer of responsibility. This means that the teacher's support (a) is tailored to students' performance level, (b) is gradually withdrawn over time depending on students' development, a fading process, which (c) is associated with gradually transferring the responsibility for learning to the students (van de Pol et al., 2010). Typical means for scaffolding are, among others, instructing, explaining, modeling, and giving hints.

A necessary condition for the effectiveness of the scaffolding approach is a coherent structure of the learning content that is accessible to the students. Instructional coherence of learning content can be characterized in three levels (Shwartz et al., 2008):

1. Learning goal coherence: learning goals are derived from a model of competence development so that they are ordered with respect to the assumed competence development of the students.
2. Intra-unit coherence: learning goals, classroom activities, and teaching practices are coordinated within a teaching unit to support students' learning.
3. Inter-unit coherence: learning goals and teaching practices are sequenced across units and years to support students' long-term development.

With respect to the topic adaptive use of strategies, it is possible to straightforwardly create a learning path based on learning goal coherence and intra-unit coherence by (a) successively introducing and practicing relevant multi-digit addition and subtraction strategies and (b) comparing the efficiency of these strategies for given problems, which includes the introduction of criteria to examine the efficiency of problem solutions (e.g., short solutions, easy calculation steps). Taking a critical perspective, the absence of learning opportunities to foster students' creativity in inventing their arithmetic strategies is apparent. Threlfall (2009) raises the concern that the explicit teaching of strategies does not provide opportunities for students to analyze problem characteristics and to creatively invent individual strategies. This might cause less stable students' skills in adaptive use of strategies (see next section).

The Implicit Approach: Without Teacher-Directed Instruction

The assumption that students can invent efficient strategies without direct instruction by a teacher can be derived from models on children's strategy development. For example, the model of the research group of Siegler and his colleagues was developed based on a series of empirical studies which were supplemented by the computer-based Strategy Choice and Discovery Simulation Model (Shrager & Siegler, 1998; Siegler, 2003). The resulting model on strategy development suggests that working on suitable arithmetic problems allows students to experience strategy execution with increasing fluency. With increasing fluency in strategy execution, more intensive metacognitive processes are possible which can stimulate the improvement of known strategies or the invention of new strategies (Shrager & Siegler, 1998). The latter encompasses monitoring the effort of solving problems with a specific strategy which results in strengthening the association between existing strategies and problem types or in a motivation to discover new strategies. According to this model, it is sufficient to expose students to sets of suitable arithmetic problems when teaching adaptive use of strategies. Explicit teaching of strategies or their adaptive use is not necessary (Siegler, 2003).

Some alternative models also suggest that explicit teaching of adaptive use of strategies is not necessary and may even be problematic. For example, Threlfall (2009) described the discovery of new strategies during problem-solving as a kind of strategy emergence. Students do not apply a ready-made schema to solve a problem. Instead, they analyze the relation of the numbers in a problem to decide the first step for a possible solution and proceed step-by-step depending on the intermediate results. Hence, a suitable strategy emerges from the interaction between problem characteristics and conceptual knowledge. The implication for teaching is that students get the opportunity to frequently experience processes of solving arithmetic problems and comparing the efficiency of the solutions (Threlfall, 2002). According to Threlfall (2009), explicit teaching of strategies and their adaptive use should be avoided because it is not creative and does not involve the analysis of problem characteristics. Instead, opportunities for repeated analysis of problem characteristics in relation to the efficiency of the solutions allow students to associate strategies with problem types (Gravemeijer, 2004). Teachers can support the process by providing opportunities to analyze problem characteristics. In a different vein, another possible approach to implicit learning is the so-called "Zahlenblickschulung," which supports students in the recognition of number patterns and numerical relationships (see the following section; Rathgeb-Schnierer & Green, 2019; Rechtsteiner & Rathgeb-Schnierer, 2017).

Both models presented above suggest that students can generate strategies on their own without an explicit introduction through the teacher. If this is the case, it might also be beneficial for the learning of adaptive use of strategies in arithmetic. In learning sciences, the so-called "generation effect" (Slamecka & Graf, 1978) describes the phenomenon that individually generated learning content is better remembered than content that is learned by demonstration. The generation effect was frequently replicated in laboratory settings for various content areas (Bertsch et al., 2007). It is explained by the assumption that cognitive processes used to generate new content in the learning process are very similar to the cognitive processes when subsequently applying this content in other problem situations (e.g., McNamara & Healy, 2000). In the context of arithmetic strategies, this means that cognitive processes of children during the generation of a new strategy to efficiently solve a given problem are similar to the cognitive processes when they are later faced with a problem of the same type and use the same strategy to solve this problem. Hence, when working on an arithmetic problem, children might remember a strategy better when they invented this strategy as an efficient solution for a similar problem on an earlier occasion.

There are plausible arguments that students can invent strategies without explicit demonstration by teachers. However, taking a critical perspective, it is not clear whether students are also able to generate complex strategies and the challenging adaptive use of strategies without teacher explanation. Since it is a necessary condition for the generation effect that students can generate the target information correctly, an ideal-typical implicit teaching approach might have its limits in case of adaptive use of strategies because complex strategies might be less available to students.

Research on the Effectiveness of Instructional Approaches

As mentioned in the introduction, research in mathematics education has repeatedly shown that learning an adaptive use of addition and subtraction strategies is challenging for elementary school students. Against this background, it is surprising that there are only a few empirical studies that examined the effectiveness of these instructional approaches applying (quasi-)experimental designs. In the following subsection, we briefly introduce empirical studies that examined features of the explicit and implicit approach in the first years of elementary school. In the subsequent subsections, we synthesize the findings of these studies with respect to three questions addressing the effectiveness of the two approaches and their strengths and weaknesses.

Studies on the Effectiveness of the Explicit and Implicit Approach

To analyze the existing evidence for the effectiveness of both instructional approaches, we considered studies that satisfied the following conditions:

- Consideration of students' adaptive use of addition and/or subtraction strategies in the first four years of elementary school,
- examination of the implementation of one or both ideal-typical approaches in settings with a high or moderate level of ecological validity[2],
- application of a (quasi-)experimental design, and
- publication in peer-review journals.

A web search revealed only a few studies with these characteristics. Though many studies address the topic of strategy use, most of them did not apply a (quasi-)experimental design or did not reach a suitable level of ecological validity. In the following, we briefly present information about five studies from the literature which differed in their goals, design, and methodology[3]. In the subsequent subsections, we analyze the findings of these studies with respect to the effectiveness of the explicit and implicit teaching approach.

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[2] High ecological validity means that the study examined an implementation in the regular mathematics classroom; moderate level means that the examined intervention can easily be transformed to an implementation for the regular classroom.
[3] All studies are more comprehensive than presented here.

The first study by Nemeth et al. (2019) applied an experimental design to compare two different implementations of the explicit approach to teach the adaptive use of subtraction strategies (see also Nemeth et al., 2021). The sample was comprised of 236 German third graders from 12 classes, and the duration of the intervention was 14 lessons in the regular mathematics classroom. In each class, the children were randomly assigned to either an interleaved condition (i.e. strategies were presented intermixed) or a blocked condition (strategies were introduced and practiced successively). In the interleaved condition, strategy comparisons were prompted (between-strategy comparison), while in the blocked condition, students were asked to reflect on the adaptivity of a specific strategy for a given subtraction task (within-strategy comparison). The groups were compared regarding the adaptive and accurate use of strategies after the intervention and in a follow-up test after five weeks.

The second study by Klein et al. (1998) was a one-year quasi-experimental study that compared two teaching approaches in ten second-grade classes (275 students) in the Netherlands (see also Blöte et al., 2001). One group followed a traditional explicit teaching approach specifically stressing one strategy (jump strategy) which was supplemented by further strategies and the comparison of strategies concerning their efficiency at the end of the school year. The other group followed a combination of an implicit and explicit teaching approach. Here, children were asked to invent strategies and encouraged to use and compare different strategies. If they were not able to invent specific strategies, the teacher explicitly presented them and allowed students to practice these strategies. After one school year, the intervention groups were compared regarding the adaptive use of strategies and the accuracy of strategy execution.

The third study by Rechtsteiner and Rathgeb-Schnierer (2017) was a one-year study in Germany with 20 less advanced first graders selected from eight classes. Twelve children attended mathematics classes that integrated the *Zahlenblickschulung* (a specific training of students' recognition of number characteristics and relations), whereas eight students attended best practice mathematics classes without *Zahlenblickschulung*. As mentioned above the *Zahlenblickschulung* can be considered as an element of the implicit approach. Applying a qualitative analysis based on interview data, the researchers identified four main types of students based on the approaches they showed when solving arithmetic problems: (a) counting strategies, (b) consistent use of procedural mastery, (c) partly basic facts with relational expertise, and (d) basic facts extended with relational expertise. These four main types gradually differentiated students' adaptive expertise after one year of teaching.

The fourth study by De Smedt et al. (2010) with 35 third graders from Belgium focused on the indirect addition strategy as a specific subtraction strategy. In an experimental design, the effects of an implicit approach (15 children) were compared with an explicit approach (20 children) concerning the frequency, accuracy, and speed of the indirect addition strategy for subtraction problems. There were five practice and three test sessions in four weeks. In each session, a set of subtraction problems was considered. One month after the last session, a retention test was administered. The explicit teaching condition involved the demonstration of the indirect addition strategy together with opportunities for children to practice this strategy. Furthermore, indirect addition was compared with direct subtraction strategies for various problems. The children in the implicit learning environment processed a large number of small-difference subtractions, suggesting indirect addition as an efficient strategy. In both groups, no conditional knowledge concerning the efficiency of a specific subtraction strategy was addressed.

In the fifth study with 73 German third graders, Heinze et al. (2018) compared the effects of an implicit approach (35 children) and an explicit approach (38 children) on students' adaptive and accurate use of strategies. The experiment was organized as a holiday course with 16 lessons based on detailed teaching scripts. The 73 children came from 17 classes, and their 179 classmates served as the control group in follow-up tests after three and eight months. Instruction in the explicit condition was based on ideas of scaffolding and instructional coherence (Heinze et al., 2020) and covered five addition and five subtraction strategies (taught in a blocked condition, cp. Nemeth et al., 2019) as well as their efficiency for given problems. The strategies[4] were introduced successively by the teacher (about 45 minutes for each strategy including exercises for the students). As soon as the children knew two strategies, additional sessions were implemented in which students were asked to solve problems and, subsequently, to compare the efficiency of their solutions in group work sessions. In the implicit condition, the children worked on specific problems which stimulated (a) the invention of individual strategies, (b) the recognition of number relations and characteristics (like in the *Zahlenblickschulung*), and (c) the association of specific strategies with problem types for which they provide an efficient solution (Heinze et al., 2018). Also, the children were repeatedly encouraged to solve given problems, to compare their solutions in small groups, and to discuss the efficiency with respect to problem characteristics.

Effectiveness of the Explicit Approach

Concerning the effectiveness of the explicit approach for teaching an adaptive and accurate use of strategies, three main results can be synthesized from the five studies. First, the findings support the theoretical assumption that instructional coherence (learning goal coherence, intra-unit, and inter-unit

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[4] Jump strategy, split strategy, compensation strategy, and simplifying strategy for addition and subtraction as well as indirect addition strategy for subtraction.

coherence) is a relevant factor for the effectiveness of the explicit approach. Findings of Klein et al. (1998) showed that a traditional explicit approach, which emphasizes one main strategy during a school year and touches other strategies only at the end of the school year, hardly supports students in using different strategies. Results from Heinze et al. (2018) and Nemeth et al. (2019) indicated that explicit teaching of different strategies in a coherent lesson sequence allows students to acquire skills for using strategies adaptively.

Second, the research results show that the explicit approach can be effective in teaching students advanced strategies. Findings of De Smedt et al. (2010), Heinze et al. (2018), and Nemeth et al. (2019) showed that third graders used the indirect addition strategy after the explicit teaching interventions. Moreover, some students from the explicit learning condition in Heinze et al. (2018) also showed the simplifying strategy (e.g., 248 + 252 = 250 + 250). However, it is important to note that only some students used advanced strategies, and this effect was not stable in the follow-up tests.

Third, providing opportunities for students to contrast strategies and their efficiency is an important factor to increase the effectiveness of the explicit approach with respect to the adaptive use of strategies. This result can be inferred from the comparison of the interleaved and the blocked condition in the Nemeth et al. (2019) study. The findings even suggest that interleaving the digit-based subtraction algorithm and the number-based subtraction strategies in an explicit teaching approach does not lead to the predominant use of the digit-based algorithm after its introduction. Hence, interleaving seems to be an effective way to prevent the decrease of students’ skills in adaptive use of strategies after the introduction of the algorithm.

Effectiveness of the Implicit Approach

One basic assumption of the implicit approach is that students can invent and adaptively use their strategies through repeated activities in analyzing arithmetic problem characteristics and solving these problems. Results from the study of Heinze et al. (2018) supported this assumption. Though the teachers from the implicit learning condition in this study only provided opportunities to analyze problems, to solve problems, and to compare the solutions, the students were able to invent a variety of individual strategies and to increase their skills in the adaptive use of these strategies. The findings of Rechtsteiner and Rathgeb-Schnierer (2017) emphasized the significant role of students’ skills in analyzing number characteristics and number relations in arithmetic problems to foster their adaptive strategy use through the implicit teaching approach.

A challenge for students who learned strategies through an implicit teaching approach is the invention of advanced strategies for multi-digit addition and subtraction. In the study of De Smedt et al. (2010), the indirect addition strategy was hardly used by children from the implicit learning environment. A similar tendency was reported in Heinze et al. (2018).

Comparisons of the Effectiveness of the Explicit and Implicit Approach

To our knowledge, a direct comparison of the effectiveness of the explicit and implicit teaching approach with respect to elementary students' adaptive use of strategies was only examined in the studies of De Smedt et al. (2010) (restricted to the indirect addition strategy) and Heinze et al. (2018). De Smedt et al. (2010) found that children from the explicit learning environment used the indirect addition strategy more often than students from the implicit learning environment. However, it should be mentioned that children from the explicit condition also used the challenging indirect addition strategy only with a low frequency, although they got opportunities to contrast this strategy with direct subtraction strategies. As mentioned above, the intervention of De Smedt et al. (2010) did not include the explicit teaching of conditional knowledge that provides information about which type of problems the indirect addition strategy is efficient.

In the study of Heinze et al. (2018), students in both conditions got opportunities to learn different strategies. Their results indicate that children's performance in the adaptive and accurate use of strategies did not significantly differ after the intervention. Both groups showed a strong increase in comparison to the control group. However, there were two specific differences between the explicit and implicit group. First, students taught with the implicit approach used specific strategies (e.g., the compensation strategy) more frequently in the follow-up test than their counterparts from the explicit teaching approach. This finding supports the expected generation effect that individually generated learning content is better remembered than content that is learned by demonstration. Second, students taught by the explicit approach used advanced strategies (simplifying, indirect addition) more frequently than the children from the implicit approach. This result supports the assumption that it might be too challenging for third graders to invent some of the advanced strategies. In both groups, students' skills in adaptive use of strategies decreased in the follow-up test after eight months. In particular, most students predominately used the digit-based algorithms which were introduced after the intervention. This occurs also in the explicit approach which is consistent with the findings for the blocked condition in the study of Nemeth et al. (2019).

Discussion and Implications for Future Research

The adaptive use of strategies is challenging learning content for elementary school students (e.g., Csíkos, 2016; Hickendorff, 2020; Torbeyns et al., 2009; Torbeyns & Verschaffel, 2016). Accordingly, there is a need for systematic research on instructional approaches to foster students' adaptive use of strategies.

In this article, we examined the explicit and implicit teaching approach – two ideal-typical instructional approaches that represent pedagogically distinct origins (teacher-directed instruction vs. student-initiated inventions). In the conceptual literature, we found supporting arguments but also possible limitations for both approaches. The few empirical studies from which we can infer information about the effects of these approaches are mostly restricted to specific elements of the instructional approaches. Nevertheless, the empirical evidence supports the assumption that both teaching approaches (or specific elements of these approaches) are effective in supporting elementary school students in the acquisition of addition and subtraction strategies as well as the adaptive and accurate use of these strategies. The studies also suggest that students' ability to invent strategies might be limited regarding some advanced strategies. This result seems to be a disadvantage to the implicit approach. However, Heinze et al. (2018) found that children participating in either of the two teaching approaches reached a similar level of skills in adaptive use of strategies for given problems. The restricted strategy repertoire of the children from the implicit approach turned out to be sufficient for finding efficient strategies for the presented problems. Finally, the studies suggest that the explicit teaching approach seems to be inferior to the implicit approach with respect to the long-term effects on students' skills in the adaptive use of strategies. This result might be explained by the generation effect, which is only expected for students taught by the implicit approach.

The relative lack of empirical studies underlines the need for more (quasi-) experimental research on teaching adaptive use of strategies. First, there is a need for replication studies to confirm the empirical findings presented in the previous section. Second, to provide information for a possible generalization of the current findings, there is a need for an extension of the previous studies through variation of different study characteristics. This might encompass, for example, the examination of both teaching approaches with (a) samples of different characteristics (e.g., variation in age, focus on low-performing students), (b) the length of the treatment, or (c) implementation in different classroom cultures. Based on the existing findings, future researchers might consider the following empirical questions as next steps:

- What are effective ways to improve the implicit teaching approach? The qualitative study of Rechtsteiner and Rathgeb-Schnierer (2017) gave some insights into the important role of students' skills in recognizing problem characteristics and number relations. How can corresponding learning environments (like the *Zahlenblickschulung*) be optimized? Which sequence of addition and subtraction problems provides effective learning opportunities for the invention of new strategies? Is it possible

to develop special addition and subtraction problems that promote the generation effect? How can students be supported in inventing advanced strategies like indirect addition?
- What are the effects of the explicit and implicit approach on students' adaptive use of strategies in longitudinal comparison studies? The interventions of De Smedt et al. (2010) and Heinze et al. (2018) "only" covered 2-3 weeks. What happens if we teach students through an explicit or implicit approach for the first three years in elementary school? Will the effects of the strengths and weaknesses of the approaches be stronger? Will differences between the effects of the approaches disappear?
- Which kind of students' prerequisites is relevant for the effectiveness of the different teaching approaches? According to the literature, conceptual knowledge of numbers is highly relevant (Baroody, 2003; Rathgeb-Schnierer & Green, 2019; Threlfall, 2002). What about other cognitive and affective prerequisites like, for example, students' working memory capacity? If students can process many calculation steps mentally due to a large working memory capacity, then they might not be motivated to learn shortcut strategies?

Finally, from the perspective of educational practice, it might be interesting to investigate the explicit and implicit approach with respect to effective arithmetic teaching in elementary school. Relevant questions for this type of studies might be:

- What would be a clever combination of explicit and implicit teaching elements? How effective would such an approach be on students' development? In the study of Klein et al. (1998), a combined teaching approach was more effective in promoting children's adaptive use of strategies than a traditional explicit approach emphasizing only one strategy. However, we need a more detailed analysis of the effects by comparing such a combined approach with implementations following ideal-typical explicit and implicit approaches. Especially important is determining whether the integration of explicit elements in the implicit teaching approach decreases students' creativity (e.g., Threlfall, 2009).
- What is the role of the digit-based algorithms in relation to the number-based calculation strategies? In a follow-up test, most students in the Heinze et al. (2018) study predominately used the standard algorithms which were introduced some months after the intervention. There was no long-term effect on the teaching approach. The findings of Nemeth et al. (2019) suggested that interleaving the digit-based and the number-based strategies in an explicit teaching approach can prevent a decrease in the adaptive use of strategies. Is it possible to implement interleaving as an

implicit approach, and in which way are elementary school students able to invent the digit-based algorithms for addition and subtraction?

There are other important research questions concerning the teaching of arithmetic strategies. For example, longitudinal studies can examine in which way students' skills in adaptive use of addition and subtraction strategies influence the corresponding skills for multiplication and division. Moreover, it would be interesting to know whether elementary students' cognitive flexibility in using arithmetic strategies is beneficial for their future mathematics learning. Future studies will help us refine our models and better understand how students can be supported in learning adaptive use of strategies.

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