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Examining the Impact of a Mental Computation Classroom Intervention on the Relational Thinking of Seventh-Grade Students

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The present study examined the impact of a mental computation intervention on the relational thinking and equivalence knowledge of seventh-grade students. The instructional intervention focused on classroom discussions, the sharing of mental computation strategies, and conceptual scaffolding from the teacher. A multiple baseline design was used with 66 students in three classes. In each class, students were assessed at five time points on (a) mental computation, (b) mathematical equivalence problem solving, and (c) relational thinking, which was assessed by asking students to make judgments about true-false number sentences. Students in one class improved on equivalence problem solving after the intervention, although ceiling effects mitigated the observation of potential instructional benefits. All students improved their relational thinking skills, and performance paralleled observed gains in mental computation in two of the three classes. Relational thinking performance levels were maintained for up to 12 weeks. The link between mental mathematics and relational thinking implies that mental computation should play a more prominent role in the seventh-grade mathematics classroom.

Keywords: Mental computation instruction, relational thinking, classroom discourse, algebra

Mental computation has been described as the act of performing calculations through a sequence of numerical transformations that renders problems manageable for the solver (Jordan et al., 2010; Maclellan, 2001; Reys, 1984; Threlfall, 2002). Although external tools, such as paper and pencil, can reduce cognitive load during mental computation, the defining processes hinge on the internal manipulation of quantities whether or not external supports are used (Sowder, 1988; Torbeyns & Verschaffel, 2016). The benefits of mental computation have been well documented: It can promote conceptual understanding, enhance number and operation sense, generate flexible strategies for problem solving, support estimation and everyday decision making, and promote transfer to other mathematical domains (e.g., Carvalho & da Ponte, 2019; Gürbüz & Erdem, 2016; Markovits & Sowder, 1994; Reys et al., 1995).

Carvalho and da Ponte (2017) claimed that the study of mental computation provides a window into students' relational thinking in mathematics because both forms of mathematical activity involve reasoning about quantitative relationships and arithmetical properties. Carvalho and da Ponte also claimed that relational thinking provides critical conceptual supports for students' mental computation. Our research focused on investigating this claim from the opposite direction, namely the effect of mental computation on relational thinking. In line with Carpenter et al. (2005), we view relational thinking as a form of algebraic reasoning because it relies on "generalized arithmetic," one of the central features of algebra (Kaput, 1998). In our view, any exposure to algebraic thinking in the early years has the potential to render formal algebra instruction at the middle- and secondary levels meaningful, flexible, and connected (Knuth et al., 2016; Schliemann et al., 2003).

In this paper, we present an extension to a previous study by Kindrat and Osana (2018), who documented the impact of a classroom intervention in mental computation on the relational thinking of seventh-grade students. From a theoretical perspective, the current research promises to extend our knowledge of the interrelationships between mental computation and relational thinking, which are currently not well understood. Our work also provides useful and empirically-validated pedagogical techniques for teachers of mathematics who focus on meaning and understanding in algebra.

Theoretical Connections between Mental Computation and Relational Thinking

Jacobs et al. (2007) defined relational thinking as a form of reasoning that entails looking at numbers and expressions holistically and noticing quantitative relations among them. The central feature of relational thinking hinges on understanding the meaning of the equal sign (Freiman & Lee, 2004; Knuth et al., 2006; Molina et al., 2008). For example, when faced with the problem $25 + 17 = 22 + \square$, students who understand the equal sign to mean "the same as" could solve it via computation (i.e., $25 + 17 – 22$) or by thinking *relationally*, which would entail examining the relationship between the amounts on both sides of the equation (i.e., "25 is 3 more than 22, so the answer must be 20 to balance it out; I would need a number 3 more than 17."). Rather than "calculate an answer" (i.e., "computational thinking"; Stephens, 2006), students who engage in relational thinking focus on relations between numbers and expressions (Empson et al., 2011; Stephens & Ribeiro, 2012).

Furthermore, students who think relationally engage in mathematical transformations that are justified, often implicitly, by properties of whole number operations. For example, when asked to think relationally about 99 × 3, students can and often do (e.g., Ambrose et al., 2003) use the distributive property by transforming 99 into (100 − 1) so the product can then be computed by subtracting 3 from 300 (i.e., 99 × 3 = (100 − 1) × 3 = 300 − 3 = 297). Such transformations can also be viewed as *substitutions* – substituting 99 for (100 − 1), in this example – a key aspect of mathematical equivalence (Jones et al., 2012). In other words, each transformation by substitution results in an expression that is mathematically *equivalent* to the first (i.e., 99 × 3 is equivalent to (100 − 1) × 3, which is in turn is equivalent to 300 − 3). In sum, relational thinking can be characterized by processing quantitative relationships in ways that rely on mutually-dependent applications of equivalence concepts and mathematical properties (Stephens & Ribeiro, 2012).

Descriptive accounts of mental computation also focus on processes of transformation: A child faced with a problem to compute mentally must first decide how to transform the numbers in ways that will make it manageable. The transformation is often achieved through implicit or explicit knowledge of number properties (Thompson, 2010). Faced with 113 − 30, a child could decide to transform 30 into 10 + 10 + 10 because removing 10 three times is more manageable than removing 30 at once. Maclellan (2001) argued that mental computation entails transforming expressions into others that look different *but are not changed in value* (our emphasis). In this way, a child performing 113 − 30 is essentially relying simultaneously on processes of transformation and the concept of sameness: 113 − 30 = 113 − (10 + 10 + 10) = 113 − 10 − 10 − 10. In line with contemporary accounts of mental computation (e.g., Gürbüz & Erdem, 2016; Proulx, 2013), we argue that choosing a mental computation strategy is an act of relational thinking because it relies on creating implicit transformations that rely squarely on equivalence.

It is worth noting that not all forms of mental computation necessarily involve relational thinking, and that the extent to which it does likely relies on a number of different factors, including the task itself (Torbyns & Verschaffel, 2016). Mentally executing the standard algorithm to solve 25 + 18, for example, without considering the actual magnitudes of the numbers in the problem shows how mental computation and relational thinking are not identical constructs. Furthermore, when used, relational thinking may be applied to different extents and in different ways. Computing 25 + 10 mentally, for example, may result in fewer transformations than 25 + 18 because of the nature of the quantitative relationships involved.

Mental Computation as a Vehicle for Enhancing Relational Thinking

Although descriptions of children's algebraic reasoning have been reported in the literature (e.g., Cai & Knuth, 2011; Kaput et al., 2008), little systematic research exists on the instructional conditions that foster relational thinking in the classroom. It has been well established that children at all levels of elementary hold deep-seated operational misconceptions about the meaning of the "=" symbol (Jacobs et al., 2007; Kieran, 1981; McNeil et al., 2017; Seo & Ginsburg, 2003; Sherman & Bisanz, 2009), particularly when asked to solve equivalence problems that are not presented in their standard, or typical, form (McNeil et al., 2006; Powell, 2012). Standard problems are open-number sentences that have one number on the right side of the equal sign. Those that deviate from this typical form have been called "non-standard" or "non-canonical" (Bisanz et al., 2009; Li et al., 2008; Powell, 2012; Rittle-Johnson et al., 2011). When asked to solve non-standard problems, such as $9 + 3 = \_\_ + 5$, for instance, children commonly add all the numbers (i.e., place 17 in the blank) or ignore the numbers to the right of the equal sign (i.e., place 12 in the blank). Critically, such misconceptions can lead to difficulties in relational thinking more broadly in later grades (e.g., Hunter, 2007; Knuth et al., 2005; Matthews & Fuchs, 2020; McNeil, 2014; McNeil & Alibali, 2005; Stephens, 2007).

Aside from the apparent theoretical overlap between mental computation and relational thinking, empirical evidence exists to support our hypothesis that instruction in the former would have a positive impact on the latter. In a qualitative study on third-graders' mental computation, Pourdavood et al. (2020) described the students' intuitive use of arithmetic properties, such as the distributive property, in their mental computation strategies. Using data from classroom conversations, their study highlighted how the sharing of mental computation strategies can lead to increased awareness of number patterns and quantitative relationships, key elements of relational thinking (Carpenter et al., 2005).

In a laboratory study, Kallai et al. (2011) found that increases in computational fluency in adults (i.e., rapid double- and triple-digit addition and subtraction computations) transferred to significantly higher performance on a test that assessed algebraic computation and probability relative to a control group that received no training. Kallai et al. also found that mental computation training was responsible for greater precision in interpreting symbolic representations of number. These findings imply that mental computation allows for enhanced number sense and increased precision of numerical quantities, thereby allowing a shift from procedural approaches to computation (e.g., executing the standard algorithm mentally) to greater attention to the meanings of numbers and their properties (Greeno, 1991).

From a different angle, Powell and Fuchs (2014) focused on the factors that could explain children's difficulties in pre-algebra performance. Specifically, the authors examined the effects of second-grade children's calculation skill and word-problem solving at the beginning of the school year on their ability to solve for unknown variables in equations (e.g., $5 - x = 3$. x = __) the following spring. Although word-problem solving had stronger predictive value, the authors found that children who struggled in both calculation and word-problem solving had lower scores on their measure of pre-algebraic reasoning at the end of the year.

Together, these studies (and others, e.g., Heirdsfield, 2011) converge to suggest that fluency in mental computation has the potential to engage students in various forms of mathematical reasoning that involve number sense, flexible problem-solving strategies, algebraic thinking, and conceptual understanding. Nevertheless, the mental computation instruction implemented in some studies and the outcome measures used are in many ways incongruent with the conceptualizations of the constructs targeted in our study. For example, Kallai et al. (2011) targeted speed and accuracy (i.e., "fluency") in their mental computation instruction, and provided corrective rather than conceptual feedback to students' responses. Further, the operationalization of pre-algebraic reasoning used by Powell and Fuchs (2014) as manipulating symbols to solve for an unknown deviates from our conceptualization of relational thinking. Such discrepancies in the empirical research make it more difficult to draw conclusions about the impacts of mental computation on relational thinking, and thus provide further rationale for our research.

Present Study

Our study examines the impact of mental computation on relational thinking and equivalence knowledge at the middle-school level. Our focus was on the type of mental computation described by Torbyns and Verschaffel (2016) as involving "clever calculation methods, relying on one's understanding of the basic features of the number system" (p. 100) and quantitative transformations based on numbers and their properties. The research constitutes a replication and extension of previous work by Kindrat and Osana (2018), who found that mental computation instruction conducted in the classroom was related to more appropriate interpretations of the equal sign and higher levels of relational thinking, as assessed by students' judgments of true-false number sentences (Carpenter et al., 2003).

Two questions remain unanswered by Kindrat and Osana (2018), however. First, no measure of the students' mental computation was used to ascertain whether the classroom intervention achieved its instructional objectives. Second, because of design limitations, Kindrat and Osana were only able to look at long-term effects (i.e., four weeks) for half of the sample. The present study was designed to address this limitation by examining the robustness of long-term benefits of mental computation instruction.

We investigated the impact of mental computation on the relational thinking and equivalence knowledge of seventh-grade students through an instructional intervention that consisted of classroom discussions during which students shared strategies and the teacher provided conceptual explanations of why their strategies worked (Beishuizen, 2001). Using a multiple-baseline design in three classes, we assessed the students' equivalence problem solving and relational thinking before the intervention, immediately after the intervention, and again between 4 and 12 weeks later. We also included a measure of students' mental computation at each time point; these data not only served as a manipulation check, which would necessarily enhance the internal validity of the study, but also allowed for additional support for the effectiveness of discussion-based instruction in the development of middle-schoolers' mental computation. Finally, because random assignment to classes was not feasible, we assessed students' general knowledge of the sixth-grade mathematics curriculum as a possible statistical control for prior knowledge.

The following research question guided the study: Does a classroom-based, conceptually-grounded instructional intervention on mental computation positively impact the relational thinking and equivalence knowledge of seventh-grade students? We predicted that students' relational thinking would increase immediately after the intervention in each class and be maintained up to 12 weeks later. Our approach to the data analysis was primarily quantitative. We supplemented the statistical results with illustrations of students' responses to give a flavor of the nature of their relational thinking before and after the intervention.

Method

Participants

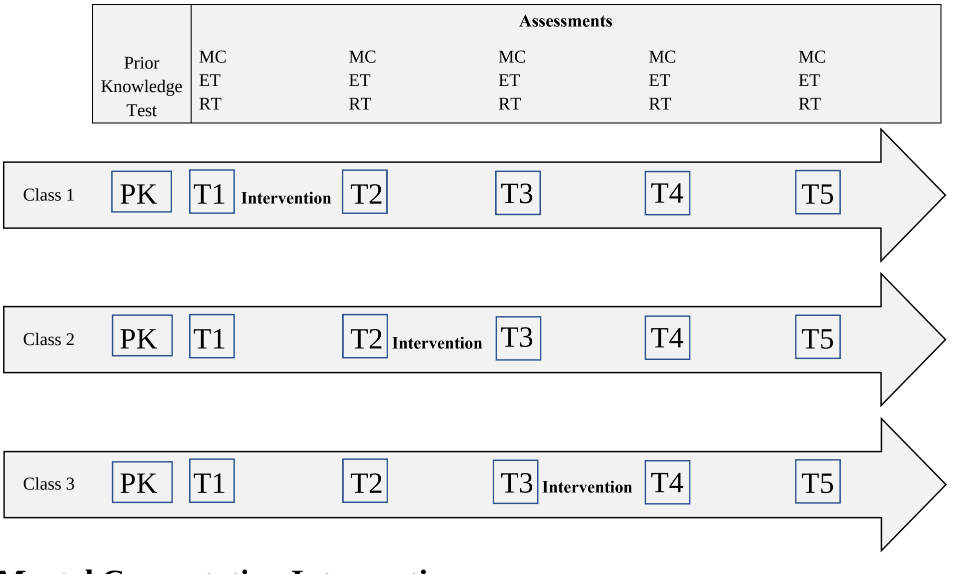
Eighty-one participants were recruited from three classes in a public suburban secondary school in the province of Quebec, Canada. The second author, who was the students' mathematics teacher in all three classes, recruited the participants by sending letters to the parents explaining the project. Fifteen students were excluded from the study: Two were transferred to another non-participating class, three did not return the consent form, two students did not receive parental consent, and the remaining eight students did not complete all of the assessments and were thus removed from the analyses. The final sample (N = 66) consisted of 23 students from Class 1, 15 students from Class 2, and 28 students from Class 3.

All three classes followed the seventh-grade mathematics program mandated by the Quebec Education Program (Gouvernement du Québec, 2020). The textbook used was *Canadian Mathematics 7* by Paholek (1993), and the topics covered during the year were decimals, whole numbers, fractions, and integer arithmetic, with no exposure to formal algebra. Students in Class 3 were enrolled in the International Baccalaureate (IB) program (https://www.ibo.org/). The IB program covered the same curriculum and schedule as the other two classes with additional practice to consolidate student learning.

Design

We employed a multiple baseline design (Levin et al., 2018), presented in Figure 1. At Time 1, tests of prior knowledge, mental computation, equivalence problem solving, and relational thinking were administered to students in all three classes. One day later, the mental computation (MC) intervention was delivered to Class 1 and subsequently to the other two classes (Classes 2 and 3) at four-week intervals. The day after each class completed the intervention (at Times 2, 3, and 4), isomorphic versions of the same measures administered at Time 1 (except for prior knowledge) were again administered to all students. Four weeks later, at Time 5, isomorphic versions of the same tests were again administered to students in all three classes to assess maintenance of mental computation, equivalence problem solving, and relational thinking.

Figure 1
Illustrations of Multiple Baseline Design



Mental Computation Intervention

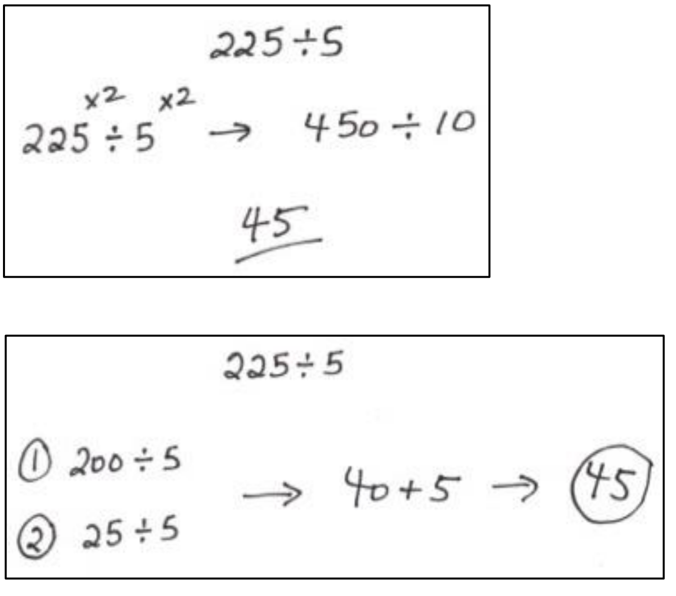
The second author, who was the students' mathematics teacher, delivered the intervention in all three classes. The intervention consisted of two types of activities for each problem. First, the teacher presented a problem on the board and the students computed the answer mentally. The second activity consisted of classroom discussions, during which the students shared their strategies with the whole class and the teacher helped the students to conceptually justify them. No data were collected during these sessions.

The intervention occurred in the first 20 minutes of each of 15 mathematics periods over four weeks. In each class period, the students mentally computed answers to six or seven problems (e.g., 48 × 5). Students were provided no materials other than individual whiteboards and dry-erase markers. The teacher wrote each problem on the board (without the equal sign) and gave the students 30 seconds to mentally compute the answer independently. After each problem, the teacher engaged the students in discussions during which they shared their strategies orally with their peers. She presented their strategies on the board but never used the equal sign in any of her representations. Figure 2 presents how the teacher might have documented the strategies of Mark and Sophie, two hypothetical students in the class, on the board. Mark explained his strategy for solving 225 ÷ 5 by saying that he first knew that 5 went into 200 forty times, and then into 25 five times, which yielded 40 + 5. Sophie shared that she multiplied both numbers by 2, which then resulted in 450 ÷ 10.

Figure 2
Teacher's Documentation of Hypothetical Student Strategies on the Board During Classroom Discussions

A

B



Note. Panel A: Mark's strategy. Panel B: Sophie's strategy. No observational data were collected during the delivery of the instructional intervention. As such, the strategies described are hypothetical and the markings made by the teacher on the board are typical of how she documented student strategies without the use of equations or the equal sign.

During classroom discussions, the teacher adhered to instructional principles documented in the literature for teaching mental computation, including allowing students the flexibility to generate their own strategies

(Heirdsfield, 2011; Threlfall, 2002) and encouraging classroom discussion on strategy use (Beishuizen, 2001; Markovits & Sowder, 1994). Specifically, the teacher focused the discussion on how the students rearranged and transformed numerical expressions to make the computations easier. The teacher's overarching goal was to provide conceptual scaffolds that supported students' understanding of how mathematical ideas and number properties could justify their strategies. For example, she explained how certain strategies might be more efficient for specific operations (e.g., dividing large numbers by factors of the divisor; multiplying the dividend and the divisor by the same factor to make the computation easier) and how the fundamental properties of arithmetic could be used to compute mentally (e.g., associative property of addition and multiplication), without explicitly naming the properties.

Prior to the start of the study, 26 sets of expressions were created for the mental computation intervention. Each set consisted of four expressions, one for each operation. The first set, for example, consisted of the expressions 62 + 38; 73 – 31; 21 × 9; and 225 ÷ 5. The second set of expressions contained the same four operations, but in a different order, namely 77 – 26; 17 × 5; 600 ÷ 4; and 42 + 58. Each subsequent set of expressions included all four operations in a different order from the previous set. On Day 1 of each class' respective MC intervention, the teacher started with Set 1 and continued through as many of the 26 sets as possible over the 15 sessions. Between 93 and 95 mental computation problems were discussed in each class across the 15-day intervention.

Measures

Prior Knowledge

The Prior Knowledge (PK) Test was a paper-and-pencil multiple-choice measure consisting of 16 items, which assessed the students' knowledge of the sixth-grade mathematics curriculum. Ten items assessed students' written computations with decimals (six items), mixed fractions of unlike denominators (two items), and exponents (two items). Two items required computations with whole numbers using order of operations, two items assessed number magnitude with decimals, and two items assess students' conversion from decimals to fractions. Prior Knowledge scores were computed by summing the total number of correct answers and converting to percent.

Mental Computation

The Mental Computation Test was designed to assess students' accuracy on problems requiring whole number mental computation. All 12 items on the test were similar to the ones covered during the intervention (e.g., 57 + 59; 107 – 38; 14 × 4; 248 ÷ 8). Problems were written on the board and read out loud; students wrote down only their final answers. Students were given 30 seconds per item and were not permitted any external tools. Mental Computation scores were the mean number of correct responses.

Outcome Measures

Equivalence Problem Solving. We administered the Equivalence Test, a paper-and-pencil measure that has been used in previous research (Kindrat & Osana, 2018; Sherman & Bisanz, 2009) to measure students' understanding of the equal sign symbol. The test consisted of 29 items with single-digit whole numbers requiring the participant to fill in the blank in an open-number sentence. Nine of the items were standard (e.g., 4 + 8 = __) and 20 were non-standard (e.g., 7 + 8 = 6 + __). Three isomorphic versions of the test, which contained different item orderings, were used for counterbalancing. Students were given 15 minutes to complete the test.

Only the responses for the 20 non-standard problems were used in the analyses. The non-standard items were designed so that incorrect answers revealed difficulty in interpreting the meaning of the equal sign rather than computational errors involving single-digit addition and subtraction. Students were permitted to use whatever strategies they wished on the test and were not instructed to use mental computation or any other approach to solve the problems. Correct answers received 1 point and incorrect answers 0 points. Equivalence scores were the mean number of correct responses.

Relational Thinking. The Relational Thinking Test (Kindrat & Osana, 2018) was used to assess the degree to which students apply relational thinking to determine the truth value of equations. The test consisted of five items. Each item consisted of a number sentence, such as 228 ÷ 6 = 456 ÷ 12, and the students were required to circle "true" or "false" and to provide a written justification for their response. Students were permitted to use their pencils, but were given no guidelines on how to think about the items, such as whether they should use mental computation or any particular strategy discussed during the intervention.

Five isomorphic versions of the Relational Thinking Test were administered at each time point (see Figure 3). In each version, the numbers used in each item were different, but the structure of the numerical relationships across versions remained the same for each operation (e.g., Version 1: 67 + 48 = 65 + 46; Version 2: 55 + 36 = 53 + 34; Version 3: 73 + 57 = 71 + 55; Version 4: 84 + 37 = 82 + 35; Version 5: 77 + 49 = 75 + 47). The students were given 20 minutes to complete the test.

Students' written justifications were coded using the following rubric (Kindrat & Osana, 2018): (a) Relational, (b) Relational with Computation, and (c) Other. Responses that were classified as Relational considered the relationship between the numbers without computing the quantities on both sides of the equal sign. Computations were included in this category only if the student had first justified the response relationally and if the computation was used to support or illustrate the relational response. Responses that were placed in the Relational with Computation category demonstrated that the student had an appropriate understanding of the equal sign, but could only judge the truth value of the equation through computation. In these cases, the student would perform the operations on both sides of the equal sign and compare the results. Responses placed in the Other category were those that displayed an operator view of the equal sign (e.g., 23 + 34 = 24 + 33 is false because 23 + 34 does not equal 24), did not supply any justification, or provided responses that were not interpretable.

Points were awarded that reflected the degree to which relational thinking was evident. Relational responses were considered optimal because they indicated that the students responded quantitatively without the need for computation. Each relational response was awarded 2 points. Relational with Computation responses were also considered relational because students showed an understanding of the equal sign, but because of the need to perform computations to determine whether the number sentence was true or false, these responses each received 1 point. All other responses (i.e., those placed in the Other category) received 0 points because they contained no evidence of relational thinking.

A Relational Thinking score was derived by computing the mean number of points per item, for a minimum of 0 and a maximum of 2 points. A random sample of 20% of the responses was coded by a second rater, and inter-rater reliability of 93% agreement was achieved. All disagreements were resolved through discussion.

Results

We first tested for pre-existing group differences (i.e., at Time 1), and then examined the impact of the instructional intervention on children's mental computation, equivalence problem solving, and relational thinking performance. Following the statistical analyses, we illustrate the nature of the students' reasoning about true-false number sentences before and after the intervention to provide a descriptive flavor of student learning.

Class Differences at Time 1

Separate one-way Analysis of Variance (ANOVA) tests were conducted on the Prior Knowledge, Mental Computation, Equivalence, and Relational Thinking measures at Time 1 to verify for any initial class differences. An alpha of .05/4 = .013 was used for each omnibus test to protect against an inflated Type I error. The classes differed on prior knowledge, $F(2, 63) = 9.35$, $p < .001$, partial eta-squared = .23, and mental computation, $F(2, 63) = 12.01$, $p < .001$, partial eta-squared = .28. Follow-up Fisher Least Significant Difference (LSD; Levin et al., 1994) comparisons on the prior knowledge measure revealed that performance in Class 3 ($M = .70$, $SD = .16$) was significantly greater than performance in Class 1 ($M = .57$, $SD = .17$, $t(63) = 2.73$, $p = .01$) and Class 2 ($M = .47$, $SD = .20$, $t(63) = 4.16$, $p < .001$), which did not differ from each other. Prior knowledge was not correlated with performance on the mental computation, equivalence, or relational thinking test immediately after the intervention in each class (i.e., Time 2 in Class 1, Time 3 in Class 2, and Time 4 in Class 3; all *p*s < .05). As such, we did not use prior knowledge as a covariate in any of the statistical analyses that follow.

Figure 3
Isomorphic Versions of the Relational Thinking Test Administered at Each Time Point

Time 1

1) 45 + 26 = 47 + 28
2) 104 – 44 = 105 – 45
3) 114 ÷ 3 = 228 ÷ 6
4) 22 x 18 = 19 x 21
5) (28 x 11) – 28 = 27 x 11

Time 2

1) 65 + 36 = 67 + 38
2) 33 x 18 = 19 x 32
3) (38 x 11) – 38 = 37 x 11
4) 105 – 45 = 106 – 46
5) 228 ÷ 6 = 456 ÷ 12

Time 3

1) 106 – 46 = 107 – 47
2) (37 x 11) – 37 = 36 x 11
3) 42 + 24 = 44 +26
4) 36 x 18 = 35 x 19
5) 225 ÷ 5 = 550 ÷ 10

Time 4

1) 144 ÷ 6 = 288 ÷ 12
2) (42 x 11) – 42 = 41 x 11
3) 25 x 16 = 17 x 24
4) 63 + 42 = 64 + 44
5) 107 – 47 = 108 – 48

Time 5

1) 125 ÷ 5 = 250 ÷ 10
2) (52 x 11) – 52 = 51 x 11
3) 36 x 16 = 17 x 35
4) 73 + 52 = 75 + 54
5) 207 – 47 = 208 – 48

The classes also differed at Time 1 on the Mental Computation measure, $F(2, 63) = 12.01$, $p < .001$, partial eta-squared = .23. Fisher LSD comparisons again indicated superior performance in Class 3 relative to Class 1, $t(63) = 4.51$, $p < .001$ and Class 2, $t(63) = 3.55$, $p = .001$, with the latter two classes not differing from each other. The classes did not differ on either the Equivalence or Relational Thinking measures at Time 1 (*p*s > .013).

Effects of Mental Computation Intervention

Descriptive statistics for the mental computation, equivalence, and relational thinking measures at each of the five time points as a function of class are presented in Table 1 and graphically in Figure 4. Below we describe the results of three separate 5(time) x 3(class) ANOVA tests. In the first analysis, we used the mental computation score as the dependent variable to determine whether the students' mental computation skills themselves improved as an outcome of the intervention. This test also served as a manipulation check to supplement our interpretations of any significant effects of the intervention on the outcome measures. We then report the two analyses of interest, which each involved using the outcome measures (i.e., Equivalence Test and Relational Thinking Test, respectively) as the dependent variable.

Table 1

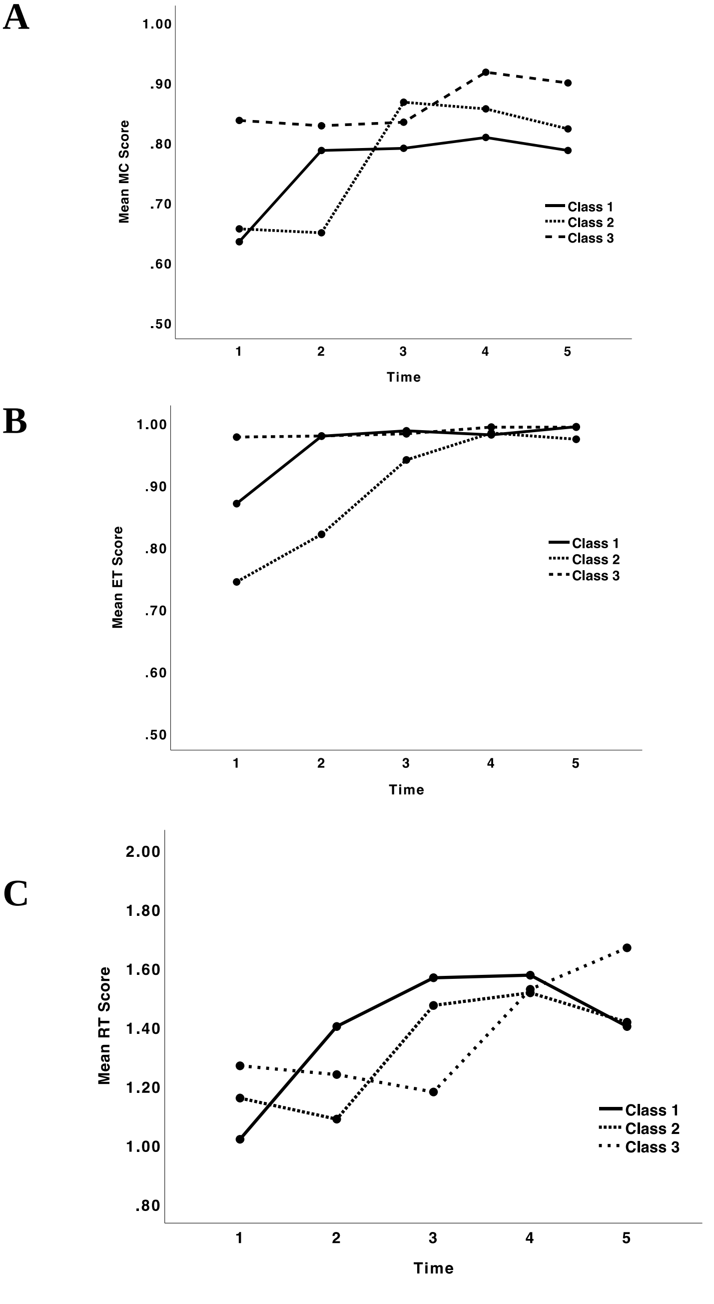
Means and (Standard Deviations) of Mental Computation, Equivalence, and Relational Thinking Tests at Each Time Point as a Function of Class

Class	*n*	Time				
		1	2	3	4	5
		Mental Computation				
1	23	.63 (.18)	.79 (.16)	.79 (.18)	.81 (.16)	.79 (.12)
2	15	.66 (.13)	.65 (.17)	.87 (.11)	.86 (.11)	.82 (.15)
3	28	.84 (.15)	.83 (.15)	.83 (.16)	.92 (.11)	.90 (.10)
		Equivalence				
1	23	.87 (.36)	.98 (.04)	.99 (.02)	.98 (.04)	.99 (.02)
2	15	.74 (.31)	.82 (.34)	.94 (.16)	.98 (.02)	.97 (.04)
3	28	.98 (.03)	.98 (.03)	.98 (.03)	.99 (.02)	.99 (.02)
		Relational Thinking[a]				
1	23	1.02 (.48)	1.40 (.30)	1.57 (.29)	1.57 (.41)	1.40 (.59)
2	14	1.16 (.33)	1.09 (.34)	1.47 (.32)	1.51 (.41)	1.41 (.36)
3	27	1.27 (.38)	1.24 (.34)	1.18 (.28)	1.53 (.36)	1.67 (.35)

[a] Minimum = 0; Maximum = 2.

Figure 4

Means of Mental Computation, Equivalence, and Relational Thinking Scores at Each Time Point as a Function of Class



Note: Panel A: Mean MC scores at each time point as a function of class. Panel B: Mean ET scores at each time point as a function of class. Panel C: Mean RT scores at each time point as a function of class. The instructional intervention was delivered between Time 1 and 2 in Class 1, between Time 2 and Time 3 in Class 2, and between Time 3 and 4 in Class 3. MC = Mental Computation; ET = Equivalence Test; RT = Relational Thinking.

Mental Computation

We were first interested in determining whether the intervention had a positive effect on students' mental computation in each class. As predicted, a time by class interaction was found, $F(8, 252) = 5.45$, $p < .001$, partial eta-squared = .15. Tests of simple effects with Bonferroni corrections revealed that the students' mental computation performance in Class 1 improved significantly immediately after the intervention, $t(252) = 4.34$, $p = .001$, and remained at the same level through all subsequent time points. In Class 2, performance improved immediately after the intervention as well, $t(252) = 5.45$, $p < .001$, and remained constant through to Time 5. Contrary to expectation, however, no improvements were observed after the intervention in Class 3, nor between Time 4 and Time 5 (all *p*s > .05).

Equivalence

On the Equivalence Test, the ANOVA revealed the expected significant time by class interaction, $F(8,252) = 3.15$, $p = .002$, partial eta-squared = .09. Improvements in performance were only seen in Class 2, however, where the mean Equivalence score was significantly higher immediately after the intervention (from Time 2 to Time 3), $t(252) = 3.63$, $p = .005$, and maintained at Time 4 and Time 5.

Relational Thinking

As predicted, a time by class interaction was revealed, $F(8,244) = 5.93$, $p < .001$. Tests of simple effects with Bonferroni corrections demonstrated an improvement from Time 1 to Time 2 in Class 1, $t(244) = 4.40$, $p < .001$, with stable performance through to Time 5 (all *p*s < .05). The same pattern was found in Class 2: Improvement was revealed from Time 2 to Time 3, $t(244) = 3.71$, $p = .004$, again with stable levels of performance through to Time 5 (all *p*s < .05). Similarly in Class 3, significant improvement was found from Time 3 to Time 4, $t(244) = 4.41$, $p < .001$, with no change from Time 4 to Time 5 ($p < .05$).

Examples of student responses on the Relational Thinking Test before and after the intervention are presented in Figure 5 (Class 1) and Figure 6 (Class 3).

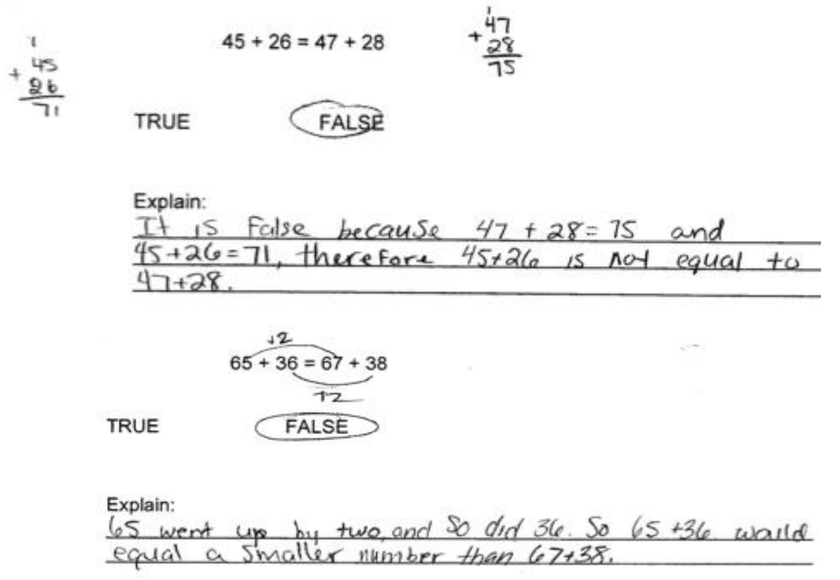
In both cases, the students used computational strategies before the intervention to assess the truth value of the number sentences provided. In contrast, immediately after the intervention in each case, the student examined the relationships between both sides of the equation without computation. The responses reveal a relational understanding of the equal sign and knowledge of how compensation works for both addition and subtraction. Figure 7 presents the response of a student from Class 3 after the intervention on the division item. In contrast to before the intervention, when the student was not able to provide any response on the division item, the student's work shows how she was able to make the connection between division and the quotient interpretation of fractions to think relationally about the quantities on both sides of the equation.

Figure 5

Sample Student Responses on the Addition Item on the Relational Thinking Test Immediately Before and After the Intervention

A

B



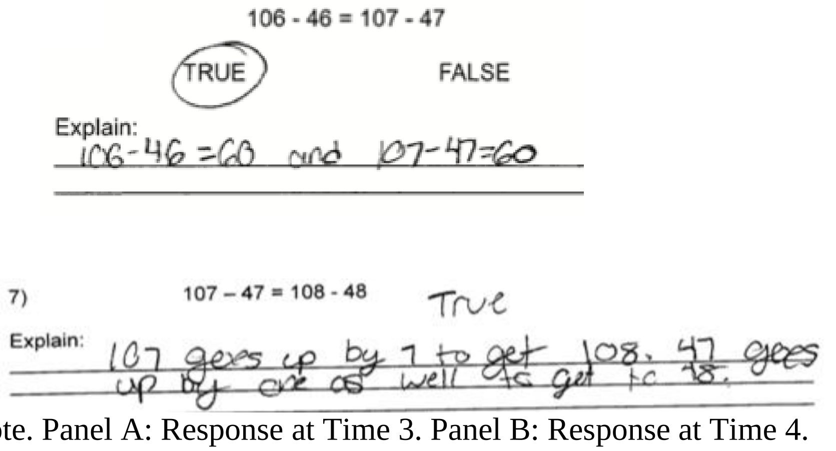
Note. Panel A: Response at Time 1. Panel B: Response at Time 2.

Figure 6

Sample Student Responses on the Subtraction Item on the Relational Thinking Test Immediately Before and After the Intervention

A

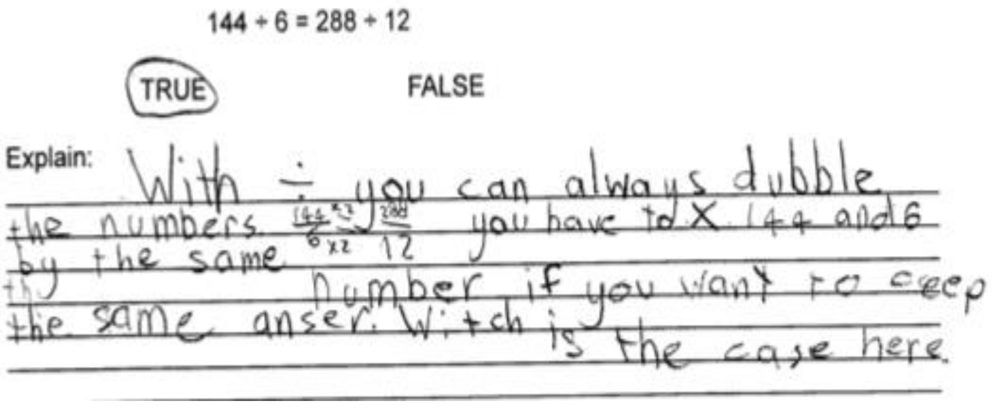
B



Note. Panel A: Response at Time 3. Panel B: Response at Time 4.

Figure 7

Sample Student Response on the Division Item on the Relational Thinking Test Immediately After the Intervention (Time 4)



Discussion

The objective of the present study was to examine the effects of a mental computation intervention on the relational thinking and equivalence knowledge of seventh graders. In three classrooms, students shared the strategies they used to solve multidigit mental computation problems, and the teacher provided conceptual scaffolds to support the comparison of different student approaches. Because of the theoretical overlaps between mental computation and algebraic reasoning (e.g., Gürbüz & Erdem, 2016) and previous empirical work in the field (e.g., Pourdavood et al., 2020), we predicted that the students would improve on their equivalence knowledge and relational thinking immediately after the intervention was delivered in each classroom.

Our hypotheses were partially supported. First, significant effects on students' equivalence problem solving performance were observed in only one of the three classes. We speculate that the ceiling effects observed on this measure, together with the relative lack of variability in performance both within and across time points, can explain the lack of growth in two of the classes. In contrast, the intervention showed the predicted effects on students' relational thinking immediately after its delivery in each class. Despite not having collected data to speak to the mechanisms involved, theoretical accounts allow us to speculate that when executing their mental computations during the intervention, the students made equivalent transformations (Jones et al., 2012) using arithmetic properties and number sense, processes that are at the heart of relational thinking (e.g., Empson et al., 2011). The students' improved mental computation performance following the intervention can also potentially account for the observed transfer to novel relational thinking contexts, in at least two of the three classes. Although the results were in the predicted direction, we speculate that no significant improvements in mental computation were found in Class 3 because of the students' relatively high level of mental computation skill before the intervention. It is possible that their greater prior knowledge in mathematics, relative to the other two classes, can at least partially explain their growth in relational thinking. Indeed, as Richland and Simms (2015) maintained, thinking relationally in mathematics relies on interconnected and flexible disciplinary knowledge.

A significant contribution of the present study is the observed direction of the relation between mental computation and relational thinking. To our knowledge, ours is the first study to show the effect in this direction using different measures for each construct. This result is particularly noteworthy given that problems such as those on the relational thinking assessment (i.e., establishing the truth value of equations) were not presented during the intervention, nor was the equal sign shown on the board or discussed at any point. A second contribution is that the students' level of relational thinking was maintained for several weeks (from 4 weeks in Class 1 to 12 weeks in Class 3), indicating potentially lasting benefits of mental computation.

The primary strength of the study lies in the internal validity of the multiple baseline design because it allowed us to rule out alternate explanations related to maturation and history (Levin et al., 2018). Another methodological strength of the study is related to the procedure we used when administering the outcome measures. On the relational thinking test, students were not reminded of the mental computation discussions that had taken place over the previous four weeks or encouraged to use mental computation or relational strategies. The finding that the students' use of relational thinking increased after the intervention, despite the possibility that they may have used computation as a less effortful strategy on the posttest, points to the power of mental computation on the development of pre-algebraic reasoning in the middle-school years.

A weakness relates to how we assessed the students' mental computation, understanding of the equal sign, and relational thinking. Although all three measures were designed to assess distinct types of knowledge, it is possible that there was overlap in the constructs tapped by the measures (e.g., mental computation used on the Relational Thinking Test and relational thinking used during mental computation), leading to potentially tautological conclusions. We argue against drawing such conclusions in this context, however, because the two measures were operationalized differently and, as such, likely assessed distinct, albeit related, skills. Precisely what processes students use on different measures of mental computation and relational thinking is a consideration for future research, without which researchers will continue to face challenges when constructing valid measures to assess them.

Other limitations are related to the incommensurate prior knowledge of the classes at the start of the study and the ceiling effects observed on the equivalence measure. Furthermore, given that arithmetic fluency is related to students' equivalence knowledge (Adrien et al., 2020), and that relational thinking is in part conceptualized by students' knowledge of equivalence (e.g., Carpenter et al., 2005), we recommend controlling for fluency in future extensions of this research (as in Liu et al., 2015) to remove any associated variance in relational thinking.

Our findings suggest that students may benefit from engaging in mental computation on a regular basis as part of their mathematical activity in the classroom. Indeed, a few minutes of mental computation incorporated into a teacher's daily routine, whether at the elementary- or middle-school level, is likely to reap long-term benefits for students' mathematical success. Nevertheless, our research cannot recommend a specific type of mental computation for the classroom or prescribe instructional approaches to support its development. Practice in mental computation may be sufficient, but alternatively, it may be the case that teachers who are skilled at directing classroom discussions that highlight the core features of relational thinking will likely have the greatest success with their students. Additional research to more precisely identify the pedagogical practices that can account for the development of students' relational thinking would be a valuable extension to the present study.

References

Adrien, E., Osana, H. P., Watchorn, R., & Bisanz, J. (2012). Individual differences in students' understanding of mathematical equivalence. In L. R. Van Zoest, J. J. Lo, & J. L. Kratky (Eds.), *Proceedings of the thirty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 997). Western Michigan University.

Ambrose, R., Baek, J., & Carpenter, T. P. (2003). Children's invention of multidigit multiplication and division algorithms. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Constructive adaptive expertise* (pp. 305-336). Erlbaum.

Beishuizen, M. (2001). Different approaches to mastering mental calculation strategies. In J. Anghileri (Ed.), *Principles and practices in arithmetic teaching* (pp. 119-130). Open University Press.

Bisanz, J., Watchorn, R. P., Piatt, C., & Sherman, J. (2009). On "understanding" children's developing use of inversion. *Mathematical Thinking and Learning*, *11*(1-2), 10-24.

Cai, J., & Knuth, E. (Eds.). (2011). *Early algebraization: A global dialogue* *from multiple perspectives*. Springer.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Heinemann.

Carpenter, T. P., Levi, L., Franke, M. L., & Zeringue, J. K. (2005). Algebra in elementary school: Developing relational thinking. *ZDM*, *37*(1), 53-59. https://doi.org/10.1007/BF02655897

Carvalho, R., & da Ponte, J. P. (2017). Mental computation with rational numbers: students' mental representations. *Journal of Mathematics Education*, *10*(2), 17-29. https://doi.org/10.26711/007577152790010

Carvalho, R., & da Ponte, J. P. (2019). Mental computation: An opportunity to develop students' strategies in rational number division. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), Proceedings of the 11th Congress of European Research in Mathematics Education. Hal. https://hal.archives-ouvertes.fr/hal-02400904/document

Empson, S. B., Levi, L., & Carpenter, T. P. (2011). The algebraic nature of fractions: Developing relational thinking in elementary school. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 409-426). Springer.

Freiman, V., & Lee, L. (2004). Tracking primary students' understanding of the equal sign. In M. Johnsen & A. Berit (Eds.), *Proceedings of the 28th international group for the psychology of mathematics education* (Vol. 2, pp. 415-422). Bergen University College.

Gouvernement du Québec. (2020). Québec Education Program: Mathematics, http://www.education.gouv.qc.ca/fileadmin/site_web/documents/education/jeunes/pfeq/PFEQ_mathematique-premier-cycle-secondaire_EN.pdf.

Greeno, J. G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education, 22*(3), 170-218. http://www.jstor.com/stable/749074

Gürbüz, R., & Erdem, E. (2016). Relationship between mental computation and mathematical reasoning. *Cogent Education, 3*(1). https://doi.org/10.1080/2331186X.2016.1212683

Heirdsfield, A. M. (2011). Teaching mental computation strategies in early mathematics. *Young Children*, *66*(2), 96-102.

Hunter, J. (2007). Relational or calculational thinking: Students solving open number equivalence problems. In J. Watson & K. Beswick (Eds.), *Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia* (pp. 421-429). Mathematics Education Research Group of Australasia.

Jacobs, V. R., Franke, M. L., Carpenter, T. P., Levi, L., & Battey, D. (2007). Professional development focused on children's algebraic reasoning in elementary school. *Journal for Research in Mathematics Education, 38*(3), 258-288. https://doi.org/10.2307/30034868

Jones, I., Inglis, M., Gilmore, C., & Dowens, M. (2012). Substitution and sameness: Two components of a relational conception of the equals sign.

Journal of Experimental Child Psychology, 113(1), 166-176. https://doi.org/10.1016/j.jecp.2012.05.003

Jordan, N.C., Glutting, J., & Ramineni, C. (2010). The importance of number sense to mathematics achievement in first and third grades. *Learning and Individual Differences, 20*(2), 82-88. https://doi.org/10.1016/j.lindif.2009.07.004

Kallai, A. Y., Schunn, C. D., Ponting, A. L., & Fiez, J. A. (2011, September 8-10). Improving foundational number representations through simple arithmetical training [Paper presentation]. Society for Research on Educational Effectiveness.

Kaput, J. J. (1998). Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K-12 curriculum. In National Council of Teachers of Mathematics & Mathematical Sciences Education Board (Eds.), *The nature and role of algebra in the K-14 curriculum: Proceedings of a national symposium*. National Academy Press.

Kaput, J. J., Carraher, D. W., & Blanton, M. L. (2008). *Algebra in the early grades*. Erlbaum.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*(3), 317-326. https://doi.org/10.1007/BF00311062

Kindrat, A., & Osana, H. P. (2018). The relationship between mental computation and relational thinking in the seventh-grade. *Fields Mathematics Education Journal, 3*(1), 1-22.

Knuth, E. J., Alibali, M. W., McNeil, N. M., Weinberg, A., & Stephens, A. C. (2005). Middle school students' understanding of core algebraic concepts: Equivalence and variable. *ZDM, 37*(1), 68-76. https://doi.org/10.1007/BF02655899

Knuth, E., Stephens, A. C., Blanton, M., & Gardiner, A. (2016). Build an early foundation for algebra success. *Phi Delta Kappan, 97*(6), 65-68. https://bit.ly/2Ww4JeG.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*(4), 297- 312. http://www.jstor.org/stable/30034852

Levin, J. R., Ferron, J. M., & Garurov, B. S. (2018). Comparison of randomization-test procedures for single-case multiple-baseline designs. *Developmental Neurorehabilitation, 21*(5), 290-311. https://doi.org/10.1080/17518423.2016.1197708

Levin, J. R., Serlin, R. C., & Seaman, M. A. (1994). A controlled, powerful multiple-comparison strategy for several situations. *Psychological Bulletin*, *115*(1), 153-159. https://doi.org/10.1037/0033-2909.115.1.153

Li, X., Ding, M., Capraro, M. M., & Capraro, R. M. (2008). Sources of differences in children's understandings of mathematical equality:

Comparative analysis of teacher guides and student texts in China and the United States. *Cognition and Instruction, 26*(2), 195-217.

Liu, A. S., Kallai, A. Y., Schunn, C. D., & Fiez, J. A. (2015). Using mental computation training to improve complex mathematical performance. *Instructional Science, 43*(4), 463-485. https://doi.org/10.1007/s11251-015-9350-0

Maclellan, E. (2001). Mental calculation: Its place in the development of numeracy. *Westminster Studies in Education, 24*(2), 145-154. https://doi.org/10.1080/0140672010240205

Markovits, Z., & Sowder, J. (1994). Developing number sense: An intervention study in grade 7. *Journal for Research in Mathematics Education, 25*(1), 4-29. https://doi.org/10.2307/749290

Matthews, P. G., & Fuchs, L. S. (2020). Keys to the gate? Equal sign knowledge at second grade predicts fourth-grade algebra competence. *Child Development, 91*(1), e14-e28.

McNeil, N. M. (2014). A change–resistance account of children's difficulties understanding mathematical equivalence. *Child Development Perspectives, 8*(1), 42-47. https://doi.org/10.1111/cdep.12062

McNeil, N. M., & Alibali, M. W. (2005). Why won't you change your mind? Knowledge of operational patterns hinders learning and performance on equations. *Child Development, 76*(4), 883-899. https://doi.org/10.1111/j.1467-8624.2005.00884.x

McNeil, N. M., Grandau, L., Knuth, E. J., Alibali, M. W., Stephens, A. C., Hattikudur, S., & Krill, D. E. (2006). Middle-school students' understanding of the equal sign: The books they read can't help. *Cognition and Instruction, 24*(3), 367-385.

McNeil, N. M., Hornburg, C. B., Fuhs, M. W., & O'Rear, C. (2017). Understanding children's difficulties with mathematical equivalence. In D. C. Geary, D. B. Berch, & K. Mann Koepke (Eds.), *Mathematical cognition and learning* (pp. 167-195). Elsevier. https://doi.org/10.1016/B978-0-12-805086-6.00008-4

Molina, M., Castro, E., & Mason, J. (2008). Elementary school students' approaches to solving true/false number sentences. *PNA, 2*(2), 75-86.

Paholek, E. (1993). *Canadian mathematics 7*. Rational Publications.

Pourdavood, R., McCarthy, K., & McCafferty, T. (2020). The impact of mental computation on children's mathematical communication, problem solving, reasoning, and algebraic thinking. *Athens Journal of Education, 7*(3), 241-254. https://doi.org/10.30958/aje.7-3-1

Powell, S. R. (2012). Equations and the equal sign in elementary mathematics textbooks. *The Elementary School Journal, 112*(4), 627-648.

Powell, S. R., & Fuchs, L. S. (2014). Does early algebraic reasoning differ as a function of students' difficulty with calculations versus word problems? *Learning Disabilities Research & Practice, 29*(3), 106-116. https://doi.org/10.1111/ldrp.12037

Proulx, J. (2013). Mental mathematics, emergence of strategies, and the enactivist theory of cognition. *Educational Studies in Mathematics, 84*(3), 309-328. https://doi.org/10.2307/43589792

Reys, R. E. (1984). Mental computation and estimation: Past, present, and future. *The Elementary School Journal, 84(5),* 546–557. https://doi.org/10.1086/461383

Reys, R. E., Reys, B. J., Nohda, N., & Emori, H. (1995). Mental computation performance and strategy use of Japanese students in grades 2, 4, 6, and 8. *Journal for Research in Mathematics Education, 26(4),* 304–326. https://doi.org/10.2307/749477

Richland, L. E., & Simms, N. (2015). Analogy, higher order thinking, and education. *Wiley Interdisciplinary Reviews: Cognitive Science, 6*(2), 177-192. https://doi.org/10.1002/wcs.1336

Rittle-Johnson, B., Matthews, P. G., Taylor, R. S., & McEldon, K. L. (2011). Assessing knowledge of mathematical equivalence: A construct-modeling approach. *Journal of Educational Psychology, 103*(1), 85-104.

Schliemann, A., Carraher, D., Brizuela, B., Earnest, D., Goodrow, A., Lara-Roth, S., & Peled, I. (2003). Algebra in elementary school. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.) *Proceedings of the joint meeting of the International Group for the Psychology of Mathematics Education and the North American Chapter of the International Group for the Psychology of Mathematics Education – Volume 4* (pp. 127-134). University of Hawai'i, Honolulu.

Seo, K., & Ginsburg, H. P. (2003). "You've got to carefully read the math sentence…": Classroom context and children's interpretations of the equals sign. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Constructing adaptive expertise* (pp. 161-187). Erlbaum.

Sherman, J., & Bisanz, J. (2009). Equivalence in symbolic and nonsymbolic contexts: Benefits of solving problems with manipulatives. *Journal of Educational Psychology, 101*(1), 88-100. https://doi.org/10.1037/a0013156

Sowder, J. (1988). Mental computation and number comparisons: Their roles in the development of number sense and computational estimation. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 182–197). National Council of Teachers of Mathematics.

Stephens, M. (2006). Describing and exploring the power of relational thinking. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia* (pp. 479-486). Mathematics Education Research Group of Australasia.

Stephens, M. (2007). Students' emerging algebraic thinking in the middle school years. In J. Watson & K. Beswick (Eds.), *Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia* (pp. 678-687). Mathematics Education Research Group of Australasia.

Stephens, M., & Ribeiro, A. (2012). Working towards algebra: The importance of relational thinking. *Revista Latinoamericana de Investigación en Matemática Educativa, 15*(3), 373-402.

Thompson, I. (2010). Getting your head around mental calculation. In I. Thompson (Ed.), *Issues in teaching numeracy in primary schools* (pp. 145-156). Open University Press.

Threlfall, J. (2002). Flexible mental calculations. *Educational Studies in Mathematics, 50*(1), 29-47. https://doi.org/10.2307/3483050

Torbeyns, J., & Verschaffel, L. (2016). Mental computation or standard algorithm? Children's strategy choices on multi-digit subtractions. *European Journal of Psychology of Education*, *31*(2), 99-116.

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